

EVEN GALOIS REPRESENTATIONS AND THE COHOMOLOGY OF $GL(2, \mathbb{Z})$

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ABSTRACT. Let ρ be a two-dimensional even Galois representation which is induced from a character χ of odd order of the absolute Galois group of a real quadratic field. After imposing some additional conditions on χ , we attach ρ to a Hecke eigenclass in the cohomology of $GL(2, \mathbb{Z})$ with coefficients in a certain infinite-dimensional vector space over a field of characteristic not equal to 2.

1. INTRODUCTION

In this paper a Galois representation will be a continuous representation $\rho : G_{\mathbb{Q}} \rightarrow GL(n, \mathbb{F})$ where \mathbb{F} is either a topological field of characteristic 0 or a finite field. When the characteristic of \mathbb{F} is not two, we say that ρ is odd if the image of complex conjugation is conjugate to a diagonal matrix with alternating 1's and -1 's on the diagonal. If \mathbb{F} has characteristic two, every Galois representation is considered to be odd. When $n = 2$, ρ is odd, and \mathbb{F} is a finite field, Serre's conjecture [13] (now a theorem of Khare and Wintenberger [10, 11]) states that ρ is attached to a modular form that is an eigenform of the Hecke operators. This means that the characteristic polynomial of the image of a Frobenius element at an unramified prime ℓ under ρ equals a certain polynomial created from the eigenvalues of the Hecke operators at ℓ . Other papers [2, 3, 9] conjecture a similar attachment for $n \geq 2$, with modular forms replaced by elements of arithmetic cohomology groups. Work of Scholze [12] proves that any eigenclass of the Hecke operators in the cohomology of a congruence subgroup of $SL(n, \mathbb{Z})$ with coefficients in a finite-dimensional admissible module M over a field \mathbb{F} has an attached Galois representation. For a field \mathbb{F} of characteristic 0, this theorem was already proven in [8] by Harris, Lan, Taylor and Thorne. Caraiani and Le Hung [5] showed that the representation guaranteed by Scholze's theorem must be odd. ("Admissible" means that if \mathbb{F} has characteristic 0 then M is an algebraic representation, and if \mathbb{F} has positive characteristic, then the matrices used to define the Hecke operators act on M via reduction modulo some fixed integer.)

In this paper, we attach certain even Galois representations to eigenclasses in arithmetic cohomology groups. The details of our main result may be seen in Theorem 11.1, at the end of the paper. Following [5] we know that we will need to use a non-admissible, infinite dimensional coefficient module for the cohomology. We also have to be careful with the exact definition of "attachment", which we now explain.

Let f be a modular form of weight $k \geq 0$ on the upper half plane, with level $\Gamma_1(N)$ and nebentype θ , and suppose that f is an eigenform for the Hecke operators

T_ℓ and $T_{\ell,\ell}$ for all $\ell \nmid N$. Denote the eigenvalue of T_ℓ by a_ℓ , and the eigenvalue of $T_{\ell,\ell}$ by A_ℓ . When $k \geq 2$, and f is holomorphic, there is a Galois representation ρ such that for all $\ell \nmid N$,

$$\det(I - \rho(\text{Frob}_\ell)X) = 1 - a_\ell X + \ell A_\ell X^2,$$

where (in this case), A_ℓ is easily seen to be equal to $\ell^{k-2}\theta(\ell)$.

The cases when $k = 0$ or 1 are different, because then f is not cohomological. If $k = 1$ and f is holomorphic, or if $k = 0$ and f is a Maass form where the eigenvalue of the Laplacian is $1/4$, there is an attached Galois representation (this is only conjectural in the Maass form case) with finite image. In both cases, the motivic weight of f is 0 , and the characteristic polynomial of Frob_ℓ equals

$$1 - a_\ell X + \theta(\ell)X^2.$$

These forms of the Hecke polynomials depend on the usual normalization of the Hecke operators.

In this paper, we will deal with the cohomology of $\text{GL}(2, \mathbb{Z})$ with a nonadmissible, infinite dimensional coefficient module and an even Galois representation. Using the same normalization of the Hecke operators as in the finite dimensional coefficient setting, we define attachment of the Galois representation as follows (compare [2, Def. 2.1]). Note that this definition of “attachment” is for the purposes of this paper only, although we may guess that it will be the correct definition to use for all even two-dimensional Galois representations.

Definition 1.1. Let V be a Hecke module over the field \mathbb{F} , and let $v \in V$ be an eigenvector for the Hecke operators T_ℓ and $T_{\ell,\ell}$ for almost all primes. Let a_ℓ be the eigenvalue of T_ℓ acting on v , and A_ℓ the eigenvalue of $T_{\ell,\ell}$ acting on v . Let $\rho : G_{\mathbb{Q}} \rightarrow \text{GL}(2, \mathbb{F})$ be a Galois representation. We say that ρ is attached to v if, for almost all ℓ ,

$$\det(I - \rho(\text{Frob}_\ell)X) = 1 - a_\ell X + A_\ell X^2.$$

Our main theorem (Theorem 11.1) then takes the following form.

Let K be a real quadratic field of discriminant d , let \mathbb{F} be a field with characteristic not equal to 2 , and let $\chi : G_K \rightarrow \mathbb{F}^\times$ be a character with finite image, satisfying certain conditions (described in Theorem 11.1). Then $\rho : G_{\mathbb{Q}} \rightarrow \text{GL}(2, \mathbb{F})$ given by $\rho = \text{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ is an even Galois representation, and is attached to a Hecke eigenclass in $H^1(\text{GL}(2, \mathbb{Z}), M_{S,q}^*)$, where q is a character related to K (see Definition 3.7), $M_{S,q}$ is defined in Definition 4.1, and the asterisk denotes \mathbb{F} -dual.

The coefficient module $M_{S,q}$ that we use is naturally defined in terms of the field K . It is a non-algebraic infinite-dimensional module somewhat related to the kind that we used in [1] to study reducible cases of the Serre-type conjecture for $\text{GL}(3)/\mathbb{Q}$.

Any Maass eigenform with eigenvalue $1/4$ is conjectured to have a Galois representation attached. Also, the Galois representations we work with in this paper are known to be attached to Maass forms. Our innovation is to prove attachment to something cohomological. Besides the intrinsic interest of this, we hope to be able to use our main theorem, combined with techniques similar to those of [1], to prove a Serre type conjecture for the sum of ρ and a character such that the three-dimensional representation as a whole is odd, in the context of $\text{GL}(3)/\mathbb{Q}$.

The idea of our proof is the following. We view K as a two-dimensional \mathbb{Q} -vector space. We construct a $\text{GL}(2, \mathbb{Q})$ module M consisting of formal sums of homothety

classes of \mathbb{Z} -lattices in K , where the homotheties are given as multiplication by the elements of a carefully chosen subgroup $K_{S,q}$ of K^\times . We use homothety classes, rather than the lattices themselves, so that the stabilizer of a homothety class in $\mathrm{GL}(2, \mathbb{Z})$ will be an infinite cyclic group generated by the image g of a unit in the ring of integers of K under an embedding of K into $\mathrm{GL}(2)$ as a non-split torus. This matrix g also stabilizes a closed geodesic in the quotient of the upper half plane modulo $\mathrm{GL}(2, \mathbb{Z})$. Our initial idea was to work with the fundamental classes of these closed geodesics, but of course we don't want to view them in the homology of $\mathrm{GL}(2, \mathbb{Z})$ with admissible coefficients for the reasons stated above. Instead we use the more algebraic approach of this paper.

We focus on the submodule $M_{S,q}$ of M which consists of formal sums that have finite support modulo the center and which have central character q . As we just said, the stabilizer of a homothety class of lattices is an infinite cyclic group. This allows us to use Shapiro's lemma to write the homology H_1 of $\mathrm{GL}(2, \mathbb{Z})$ with coefficients in $M_{S,q}$ in terms of the H_1 of these cyclic stabilizers, which is an algebraic version of the fundamental classes of the corresponding closed geodesics. We then have to understand how the Hecke operators act.

We use the method of partial Hecke operators described in [1] to get a tractable formula for the action of a Hecke operator on $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$. Now a class in that homology group has finite support (modulo the center) on chains, and the Hecke operators always expand the support. So there will not be any Hecke eigenvectors in the homology group $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$. We must seek for Hecke eigenvectors in the dual space $H^1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}^*)$. We interpret elements of the dual space as functions on the space of lattices in K . In order for us to construct such functions in a way that makes it possible to compute the Hecke operators at ℓ , we use the Bruhat-Tits graph \mathcal{T}_ℓ for $\mathrm{GL}(2, \mathbb{Q}_\ell)$ or a double cover \mathcal{T}_ℓ^2 of \mathcal{T}_ℓ , depending on whether ℓ is split or inert in K . We then relate the Hecke operators at ℓ to a Laplacian on \mathcal{T}_ℓ (or \mathcal{T}_ℓ^2) and to the action of the center. This allows us to construct lattice functions that have the desired Hecke eigenvalues.

These functions are restricted infinite products over the rational primes of the local functions we construct on the graphs. Lattices which are fractional ideals in K play a special role in the study of $M_{S,q}$ and we call them "idealistic" lattices. The construction of the local functions depends on the crucial distinction between idealistic and non-idealistic lattices. To define a cohomology class, the infinite product has to satisfy a certain global invariance property (proved in Section 10), which is guaranteed by the fact that χ can be viewed as a global character on ideals.

In case \mathbb{F} has characteristic 2, where the distinction between odd and even Galois representations breaks down, a simplified version of our construction works to attach ρ to a Hecke eigenclass in the cohomology of $\mathrm{GL}(2, \mathbb{Z})$ with coefficients in the analog of $M_{S,q}$ (where now q would be the identity central character.) In this case, we can work directly with the usual Bruhat-Tits graph for both split and inert primes. We do not cover this in this paper because when the characteristic of \mathbb{F} equals 2, ρ is deemed to be odd and is known to be attached to a homology class with admissible coefficients, by the work of Khare and Winteberger [10, 11] referred to above.

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2. LATTICES AND HOMOTHETIES IN K

Fix a real quadratic field K , its ring of integers \mathfrak{O} and an element $\omega \in \mathfrak{O}$ such that $\mathfrak{O} = \mathbb{Z}[\omega]$. Let d be the discriminant of K/\mathbb{Q} . Let ϵ be a fundamental unit, i.e. a unit whose image modulo ± 1 generates $\mathfrak{O}^\times / \{\pm 1\}$.

Consider K as a two-dimensional vector space over \mathbb{Q} . By a lattice in K , we will mean a free \mathbb{Z} -module of rank 2 contained in K . Such a module has as a \mathbb{Z} -basis two \mathbb{Q} -linearly independent elements.

Let Y be the set of all column vectors ${}^t(a, b) \in K^2$ with $b \neq 0$ and $a/b \notin \mathbb{Q}$. If we let $\bar{\omega} = {}^t(\omega, 1) \in Y$, then every element of Y is of the form $\gamma \bar{\omega}$ for some $\gamma \in \text{GL}(2, \mathbb{Q})$. In addition, given $y, y' \in Y$, there is a unique $\gamma \in \text{GL}(2, \mathbb{Q})$ with $y = \gamma y'$. There is a natural action of K^\times by scalar multiplication on Y , which we write as a right action.

Definition 2.1. Let $y = {}^t(a, b) \in Y$. Define Λ_y to be the \mathbb{Z} -lattice in K generated by a and b (i.e. the set of all integer linear combinations of a and b).

Note that for $\alpha \in K^\times$, we have $\Lambda_{y\alpha} = \alpha \Lambda_y$.

Definition 2.2. Let $H \subseteq K^\times$ be a multiplicative subgroup of K^\times . Two lattices Λ_1 and Λ_2 in K will be said to be *homothetic* if there is some $\alpha \in K^\times$ such that $\Lambda_1 = \alpha \Lambda_2$. If $\alpha \in H$, we will say that the lattices are *H-homothetic*.

Homothety and *H*-homothety of lattices are equivalence relations on the set of all lattices in K .

Definition 2.3. Let H be a multiplicative subgroup of K^\times . Define Y/H to be the quotient of Y with respect to the right action of scalar multiplication by H . The left action of $\text{GL}(2, \mathbb{Q})$ on Y then gives a left action of $\text{GL}(2, \mathbb{Q})$ on Y/H .

Lemma 2.4. *There is a bijection between $\text{GL}(2, \mathbb{Z})$ -orbits of elements of Y/H and *H*-homothety classes of lattices in K .*

Proof. Denote the set of *H*-homothety classes of lattices in K by \mathcal{H} . Define a map

$$f : Y/H \rightarrow \mathcal{H}$$

by setting $f(x)$ equal to the *H*-homothety class of Λ_y for any $y \in Y$ representing $x \in Y/H$. Since the various y representing x all differ by scalar multiples by some element of H , it is clear that f is well defined. Since every lattice in K is of the form Λ_y for some $y \in Y$, the map f is surjective. Finally, for $\gamma \in \text{GL}(2, \mathbb{Z})$, we have that γx is represented by γy , and that $\Lambda_{\gamma y} = \Lambda_{\gamma y}$. Hence, f is constant on $\text{GL}(2, \mathbb{Z})$ -orbits, and so induces a surjective map \hat{f} from the set of $\text{GL}(2, \mathbb{Z})$ -orbits in Y/H to \mathcal{H} .

Now suppose that $x, x' \in Y/H$ are represented by $y, y' \in Y$, and $f(x) = f(x')$. Then $\Lambda_y = \alpha \Lambda_{y'} = \Lambda_{y'\alpha}$ for some $\alpha \in H$. Hence, the entries of both y and $y'\alpha$ are a basis for Λ_y . Therefore, there is some $\gamma \in \text{GL}(2, \mathbb{Z})$ such that $y = \gamma y'\alpha$. Then $x = \gamma x'$, so x and x' are in the same $\text{GL}(2, \mathbb{Z})$ -orbit. Hence, \hat{f} is an injective map on $\text{GL}(2, \mathbb{Z})$ -orbits. \square

Lemma 2.5. *Let Λ be a lattice in K . Then there is a minimal positive integer m such that $\epsilon^m \Lambda = \Lambda$.*

Proof. Note that if Λ and Λ' are K^\times -homothetic, the lemma will be true for Λ if and only if it is true for Λ' , with the same value of m (since K^\times is commutative.)

Hence, we may, without loss of generality, assume that Λ is contained in \mathfrak{D} . Since Λ is a rank two \mathbb{Z} -submodule of \mathfrak{D} , it must have finite index in \mathfrak{D} . We may thus choose an $N \in \mathbb{Z}$ such that $N\mathfrak{D} \subseteq \Lambda \subseteq \mathfrak{D}$. Since $\mathfrak{D}/N\mathfrak{D}$ is finite and multiplication by ϵ permutes its elements, there is some positive $m \in \mathbb{Z}$ such that $\delta = \epsilon^m$ acts trivially on $\mathfrak{D}/N\mathfrak{D}$, and hence on $\Lambda/N\mathfrak{D}$. Then δ must take Λ to itself, so $\delta\Lambda \subseteq \Lambda$. We must also have $\delta^{-1}\Lambda \subseteq \Lambda$, so $\Lambda \subseteq \delta\Lambda \subseteq \Lambda$, and therefore $\delta\Lambda = \Lambda$. The existence of a minimal positive m satisfying the conditions of the theorem follows immediately from the existence of some positive m . \square

Definition 2.6. Given $x \in Y/H$, we define Γ_x to be the stabilizer of x in $\mathrm{GL}(2, \mathbb{Z})$, and $\hat{\Gamma}_x$ to be the quotient

$$\frac{\Gamma_x}{\Gamma_x \cap \{\pm I\}}.$$

Definition 2.7. We will say that a subgroup H of K^\times is *unit-cofinite* if $H \cap \mathfrak{D}^\times$ has finite index in \mathfrak{D}^\times .

Theorem 2.8. *Let H be a unit-cofinite subgroup of K^\times , and let $x \in Y/H$ be represented by $y \in Y$. Then $\hat{\Gamma}_x$ is a cyclic group, generated by the image of the unique element $g \in \Gamma_x$ satisfying*

$$gy = y\delta$$

where $\delta = \pm\epsilon^m$, and m is smallest possible positive integer such that $\Lambda_y = \Lambda_{y\epsilon^m}$ and $\delta \in H$.

Remark 2.9. The notation in the theorem means that $\delta = \epsilon^m$ or $\delta = -\epsilon^m$, depending on which is in H . If both are in H , then $-1 \in H$ and we set $\delta = \epsilon^m$. If $-1 \notin H$, it is possible to have $-\epsilon^m \in H$ without having $\epsilon^m \in H$. Hence, it is necessary to choose $\delta = \pm\epsilon^m$ with m minimal to get a generator of Γ_x .

Proof. Let x be represented by $y = {}^t(a, b) \in Y$. Choose the smallest positive m such that $\epsilon^m \Lambda_y = \Lambda_y$ and one (or both) of ϵ^m and $-\epsilon^m \in H$. Let $\delta = \pm\epsilon^m \in H$. Then $\delta\Lambda_y = \Lambda_y$, so $y\delta$ is a basis of Λ_y . Hence, there is some $g \in \mathrm{GL}(2, \mathbb{Z})$ such that $gy = y\delta$. Since $\delta \in H$, we see that $gx = x$.

We now show that every element in Γ_x is (up to a sign) a power of g . Let $\eta \in \Gamma_x$. Then, since $\eta x = x$, there is some $\alpha \in H$ such that $\eta y = y\alpha$. Now α is an eigenvalue of η , and $\eta \in \mathrm{GL}(2, \mathbb{Z})$, so $\alpha \in \mathfrak{D}^\times$. Hence, $\alpha = \pm\epsilon^r$. By the division algorithm and the minimality of m , we see that $\alpha = \pm\delta^k$ for some k . Hence, $\eta = \pm g^k$. If $\eta = g^k$ we are finished. If $\eta = -g^k$ then $-I \in \Gamma_x$ and $\eta \equiv g^k$ modulo $\Gamma_x \cap \{\pm I\}$. \square

Certain elements $x \in Y/H$ will be quite important to us; for these elements, the value of m in the previous proof is determined solely by H .

Definition 2.10. Let H be a multiplicative subgroup of K^\times . If $x \in Y/H$ can be represented by $y \in Y$ such that Λ_y is a fractional ideal in K , then we say that x is *idealistic*.

Note that determining whether x is idealistic does not depend on the choice of $y \in Y$ representing x .

Corollary 2.11. *Let H be a unit-cofinite subgroup of K^\times . If $x \in Y/H$ is idealistic, the value of m in Theorem 2.8 is equal to the smallest positive integer k such that $\pm\epsilon^k \in H$.*

Definition 2.12. Let H be a unit-cofinite subgroup of K^\times . For $x \in Y/H$, denote the positive integer m described in Theorem 2.8 by m_x , and the element g described in Theorem 2.8 by g_x .

Corollary 2.13. Let H be a unit-cofinite subgroup of K^\times . If $x, x' \in Y/H$ are in the same $\mathrm{GL}(2, \mathbb{Z})$ -orbit, then $m_x = m_{x'}$.

Proof. If $x = \gamma x'$ for $\gamma \in \mathrm{GL}(2, \mathbb{Z})$, then $\Gamma_{x'} = \gamma \Gamma_x \gamma^{-1}$, so g_x is conjugate to $g_{x'}$. Then g_x and $g_{x'}$ have the same eigenvalues, so x and x' have the same value of m . \square

Assume that H does not contain -1 . In this case, Γ_x does not contain $-I$, so $\hat{\Gamma}_x = \Gamma_x$ is cyclic, generated by g_x . Then, there is a canonical isomorphism

$$I_x : H_1(\Gamma_x, \mathbb{F}) \rightarrow \Gamma_x \otimes_{\mathbb{Z}} \mathbb{F}.$$

Definition 2.14. If $-1 \notin H$, and $x \in Y/H$, define z_x to be the generator of $H_1(\Gamma_x, \mathbb{F})$ such that $I_x(z_x) = g_x \otimes 1$.

3. (S, q) -HOMOTHEITIES

From now on, we fix a field \mathbb{F} of characteristic not equal to 2.

Definition 3.1. Let $y = {}^t(a, b) \in Y$. We define an injective homomorphism $r_y : K^\times \rightarrow \mathrm{GL}(2, \mathbb{Q})$ by

$$r_y(c) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac \\ bc \end{pmatrix}$$

for $c \in K^\times$.

Definition 3.2. Let M be any positive integer. Define $S_0(M)$ to be the largest subgroup of $\mathrm{GL}(2, \mathbb{Q})$ that can be mapped modulo M to $\mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z})$. Define $S(M)$ to be the kernel of reduction modulo M from $S_0(M)$ to $\mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z})$.

We note that for any M , $S_0(M)$ contains $\mathrm{GL}(2, \mathbb{Z})$.

Definition 3.3. Recall that $d = \mathrm{disc}(K)$. If the characteristic of \mathbb{F} is nonzero, set p equal to the characteristic of \mathbb{F} ; otherwise, set $p = 1$. Fix positive integers M, N with $M \geq 3$ such that $M \mid pdN$. Define $S_0 = S_0(pdN)$ and $S = S(M) \cap S_0(pdN)$.

Since S is the kernel of the composition

$$S_0 \rightarrow \mathrm{GL}(2, \mathbb{Z}/pdN\mathbb{Z}) \rightarrow \mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z}),$$

we see that S has finite index in S_0 . Further, since $M \geq 3$, $-I \notin S$.

Definition 3.4. Recall that $\bar{\omega} = {}^t(\omega, 1) \in Y$. Define

$$K_{S_0} = \{c \in K^\times : r_{\bar{\omega}}(c) \in S_0\}$$

and

$$K_S = \{c \in K^\times : r_{\bar{\omega}}(c) \in S\}.$$

We note that K_{S_0} and K_S are multiplicative subgroups of K^\times .

Since $S \subseteq S_0$ has finite index, it is clear that $K_S \subseteq K_{S_0}$ with finite index.

Lemma 3.5. If $\alpha \in K_{S_0}$, then α is relatively prime to pdN .

Proof. Since $\alpha \in K$, we may write $\alpha = \beta/n$ with $\beta \in \mathfrak{O}$ and $n \in \mathbb{Z}$. Since $\beta \in \mathfrak{O}$, we see that $r_{\bar{\omega}}(\beta)$ has integer entries. If a prime $b \mid dN$ and $b \mid n$, then, since the entries of $r_{\bar{\omega}}(\alpha) = \frac{1}{n}r_{\bar{\omega}}(\beta)$ can have no denominators divisible by b , it must be the case that b divides every entry of $r_{\bar{\omega}}(\beta)$. This implies that $b \mid \beta$ in \mathfrak{O} . Canceling (repeatedly, if needed), we may take n to be relatively prime to pdN .

Now $N_{\mathbb{Q}}^K(\alpha) = \det(r_{\bar{\omega}}(\alpha)) = \det(r_{\bar{\omega}}(\beta))/n^2 = N_{\mathbb{Q}}^K(\beta)/n^2$ must be relatively prime to pdN , so $N_{\mathbb{Q}}^K(\beta)$ is also relatively prime to pdN . Since $\beta \in \mathfrak{O}$, it must then be relatively prime to pdN , and so α is as well. \square

Lemma 3.6. K_S is a unit-cofinite subgroup of K^\times .

Proof. We note that $r_{\bar{\omega}}(\epsilon) \in \mathrm{GL}(2, \mathbb{Z})$. Hence, $r_{\bar{\omega}}(\epsilon)$ can be reduced modulo M to give a matrix in $\mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z})$. Since $\mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z})$ is finite, there is some positive integer k such that

$$r_{\bar{\omega}}(\epsilon)^k = r_{\bar{\omega}}(\epsilon^k) \in S(M).$$

Since $r_{\bar{\omega}}(\epsilon)$ is also in S_0 , we see that $\epsilon^k \in K_S$, so K_S is unit-cofinite. \square

Definition 3.7. Let Z denote the set of scalar matrices $\zeta_r = rI$ where $r \in \mathbb{Q}^\times \cap K_{S_0}$. Let $\theta : \mathbb{Z} \rightarrow \mathbb{F}^\times$ be the quadratic Dirichlet character cutting out K , and define $q : Z \rightarrow \mathbb{F}^\times$ by $q(rI) = \theta(r)$ for $r \in \mathbb{Z} \cap K_{S_0}$, extended multiplicatively to Z .

We define $q^* : K_{S_0} \rightarrow \mathbb{F}^\times$ to be the composition of the following multiplicative maps:

- (1) The map taking $a \in K_{S_0}$ to the principal fractional ideal $(a) \subset K$,
- (2) The map taking a fractional ideal to its prime factorization,
- (3) The map taking a product of powers of prime ideals to the subproduct of powers of inert prime ideals,
- (4) The map taking an inert prime ideal (ℓ) to $q(\ell I)$.

We note that q^* is a homomorphism, and we define $K_{S,q}$ to be the kernel of $q^*|_{K_S}$. Then $K_{S,q}$ is a multiplicative subgroup of K^\times and $K_{S,q}$ has index 2 in K_S .

Lemma 3.8. $K_{S,q}$ is a unit-cofinite subgroup of K^\times .

Proof. This is true because K_S is unit-cofinite, and any unit in K_S is in the kernel of q^* , and so in $K_{S,q}$. \square

Lemma 3.9. For $r \in \mathbb{Q}^\times \cap K_{S_0}$, $q(rI) = q^*(r)$.

Proof. Let

$$r = \prod_{\ell \text{ prime}} \ell^{n_\ell}.$$

Then

$$q(rI) = \theta(r) = \prod_{\ell \text{ prime}} \theta(\ell)^{n_\ell} = \prod_{\ell \text{ inert in } K} \theta(\ell)^{n_\ell} = \prod_{\ell \text{ inert in } K} q(\ell I)^{n_\ell} = q^*(r).$$

since $n_\ell = 0$ for ℓ ramified in K/\mathbb{Q} , and $\theta(\ell) = 1$ for ℓ split in K . \square

Let $X_{S,q} = Y/K_{S,q}$. Then $\mathbb{F}Y$ is an $(S_0, K_{S,q})$ -bimodule, and we obtain an isomorphism

$$\mathbb{F}Y \otimes_{K_{S,q}} \mathbb{F} \cong \mathbb{F}X_{S,q} \otimes \mathbb{F} = \mathbb{F}X_{S,q}$$

of S_0 -modules.

Lemma 3.10. Let $x \in X_{S,q}$ be represented by $y \in Y$. Then

- (a) $\Gamma_x = \{r_y(c) : c \in K_{S,q}\} \cap \text{GL}(2, \mathbb{Z})$.
- (b) If $g \in \Gamma_x$, then $g = r_y(c)$ for some $c \in \mathfrak{O}^\times$.
- (c) $-I \notin \Gamma_x$ and Γ_x is infinite cyclic.

Proof. (a) Suppose $g \in \Gamma_x$. Then we have that $gy = yc$ for some $c \in K_{S,q}$. Since $yc = r_y(c)y$, and the entries of y are a \mathbb{Q} -basis of K , we see that $g = r_y(c)$. Hence, g is in the given intersection, and any g in the given intersection fixes x .

(b) Let $g \in \Gamma_x$. Then $g = r_y(c)$, for some $c \in K_{S,q}$, and the characteristic polynomial of g is the same as that of multiplication by c on K . Since $g \in \text{GL}(2, \mathbb{Z})$, we see that $c \in \mathfrak{O}^\times$.

(c) This follows from Theorem 2.8, Lemma 3.8, and the fact that $-1 \notin K_{S,q}$. \square

Denote by $\tilde{\omega}$ the image in $X_{S,q}$ of $\tilde{\omega}$.

Definition 3.11. Define $i^S = m_{\tilde{\omega}}$.

Lemma 3.12. Let $x \in X_{S,q}$.

- (i) For any x , $i^S \mid m_x$.
- (ii) If x is idealistic, then $m_x = i^S$.

Proof. (i) For any $\xi \in X_{S,q}$, let $\phi_\xi : \Gamma_\xi \rightarrow K_{S,q}$ be the injective homomorphism defined by $g = r_\xi(\phi_\xi(g))$ for $g \in \Gamma_\xi$. Then the image of ϕ_ξ is generated by ϵ^{m_ξ} .

Now let $x \in X_{S,q}$ be represented by $y \in Y$. We have seen that any element of Γ_x is of the form $r_y(c)$ for some $c \in \mathfrak{O}^\times$. From Lemma 3.10, we see that for $c \in \mathfrak{O}^\times$,

$$r_y(c) \in \Gamma_x \implies c \in K_{S,q} \iff r_{\tilde{\omega}}(c) \in \Gamma_{\tilde{\omega}}.$$

Hence, the image of ϕ_x is contained in the image of $\phi_{\tilde{\omega}}$. Since both images are cyclic groups, we see that $i^S \mid m_x$.

(ii) Since x is idealistic, Λ_y is a fractional ideal of K . Hence, $r_y(c) \in \text{GL}(2, \mathbb{Z})$ for all $c \in \mathfrak{O}^\times$. Therefore, we see that for $c \in \mathfrak{O}^\times$,

$$r_y(c) \in \Gamma_x \iff c \in K_{S,q} \iff r_{\tilde{\omega}}(c) \in \Gamma_{\tilde{\omega}}.$$

Clearly, then, $m_x = i^S$. \square

Definition 3.13. For $x \in X_{S,q}$, set $m'_x = m_x / i^S$.

Finally, we prove the following lemma about the relationship between elements of Z and elements of $K_{S,q}$.

Lemma 3.14. Let $\zeta = rI \in \mathbb{Z}$ with $r \in \mathbb{Q}^\times \cap K_{S_0}$, let $\alpha \in K_{S,q}$, and let $y \in Y$. If $\zeta r_y(\alpha) \in \text{GL}(2, \mathbb{Z})$, then $q(\zeta) = 1$.

Proof. We have that $\zeta r_y(\alpha) = r_y(r\alpha) \in \text{GL}(2, \mathbb{Z})$. Since the characteristic polynomial of $r\alpha$ is the same as the characteristic polynomial of $r_y(r\alpha)$, we see that $r\alpha$ is a unit in \mathfrak{O} . Hence, $q^*(r\alpha) = 1$. Since $q^*(\alpha) = 1$, by the multiplicativity of q^* we see that $q^*(r) = 1$. By Lemma 3.9, it follows that $q(rI) = q(\zeta) = 1$. \square

4. DEFINING THE COEFFICIENT MODULE $M_{S,q}$

Definition 4.1. Define $M_{S,q}$ to be the \mathbb{F} -vector space of formal sums

$$\sum_{x \in X_{S,q}} c_x x$$

with $c_x \in \mathbb{F}$, such that the sum is supported on a finite number of Z -orbits of $X_{S,q}$, and such that the coefficients satisfy the relation

$$c_{\zeta x} = q^{-1}(\zeta)c_x$$

for all $\zeta \in Z$.

In this paper q has order 2, but we write q^{-1} as a check on our computations and for possible generalizations to characters of higher orders.

Definition 4.2. Define $Z_{S,q} = \{\zeta_r \in Z : r \in K_{S,q}\}$.

Note that $Z_{S,q}$ is a subgroup of finite index in Z (since $K_{S,q}$ has finite index in K_{S_0}). Let \mathcal{B} be a collection of coset representatives of $Z_{S,q}$ inside Z .

Now $Z \mathrm{GL}(2, \mathbb{Z})$ is a group which acts on $X_{S,q}$. Hence, we may choose a collection of representatives of the $Z \mathrm{GL}(2, \mathbb{Z})$ -orbits in $X_{S,q}$. We will denote such a collection by \mathcal{A} .

Note that given any $x \in X_{S,q}$, we may assume (possibly by changing \mathcal{A}) that $x \in \mathcal{A}$. For the remainder of this section, we fix a set \mathcal{A} of $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit representatives.

Clearly each Z -orbit in $X_{S,q}$ contains at least one element of the form gx with $g \in \mathrm{GL}(2, \mathbb{Z})$ and $x \in \mathcal{A}$.

Definition 4.3. Let \mathcal{C} be a collection of representatives of the Z -orbits in $X_{S,q}$, chosen so that each representative in \mathcal{C} is of the form gx for $g \in \mathrm{GL}(2, \mathbb{Z})$ and $x \in \mathcal{A}$.

Remark 4.4. Note that \mathcal{C} is not uniquely determined by \mathcal{A} . However, once a choice of \mathcal{C} is fixed, any Z -orbit will contain a unique representative $gx \in \mathcal{C}$, and the element $x \in \mathcal{A}$ and the coset $g\Gamma_x$ of $g \in \mathrm{GL}(2, \mathbb{Z})$ are uniquely defined.

For the remainder of this section, we fix a choice of \mathcal{C} corresponding to our choice of \mathcal{A} .

Lemma 4.5. *The set*

$$\left\{ \sum_{\zeta \in \mathcal{B}} q^{-1}(\zeta) \zeta x : x \in \mathcal{C} \right\}$$

is an \mathbb{F} -basis of $M_{S,q}$.

Proof. For any $x \in X_{S,q}$, the Z -orbit of x is equal to the set $\{\zeta x : \zeta \in \mathcal{B}\}$. By the relation on the coefficients of an element in $M_{S,q}$, the coefficient of ζx is equal to $q^{-1}(\zeta)$ times the coefficient of x . Since an element of $M_{S,q}$ is supported on finitely many Z -orbits, the lemma follows. \square

Lemma 4.6. *$M_{S,q}$ is an S_0 -module. For $r \in \mathbb{Q}^\times \cap K_{S_0}$, the action of ζ_r on $M_{S,q}$ is via the scalar $q(\zeta_r)$.*

Proof. For $s \in S_0$, we have $s \sum c_x x = \sum c_x s x \in M_{S,q}$, since Z is in the center of S_0 .

It suffices to prove the statement about the action of ζ_r on basis elements of the form

$$\sum_{\zeta_\ell \in \mathcal{B}} q^{-1}(\zeta_\ell) \zeta_\ell x,$$

with $x \in \mathcal{C}$. For ζ in Z , we will define $\bar{\zeta}$ to be the unique element of \mathcal{B} such that $\zeta \in \bar{\zeta}Z_{S,q}$. The map from \mathcal{B} to \mathcal{B} given by $\zeta \mapsto \bar{\zeta}\zeta_r$ for a fixed ζ_r is a bijection. We also note that $q(\bar{\zeta}) = q(\zeta)$ for any $\zeta \in Z$.

Setting $u = r\ell$, we now have

$$\begin{aligned}
\zeta_r \sum_{\zeta_\ell \in \mathcal{B}} q^{-1}(\zeta_\ell) \zeta_\ell x &= \sum_{\zeta_\ell \in \mathcal{B}} q^{-1}(\zeta_\ell) \zeta_\ell \zeta_r x \\
&= \sum_{\zeta_\ell \in \mathcal{B}} q^{-1}(\zeta_\ell) \zeta_{\ell r} x \\
&= \sum_{\bar{\zeta}_u \in \mathcal{B}} q^{-1}(\zeta_u \zeta_r^{-1}) \zeta_u x \\
&= q^{-1}(\zeta_r^{-1}) \sum_{\bar{\zeta}_u \in \mathcal{B}} q^{-1}(\bar{\zeta}_u) \bar{\zeta}_u x \\
&= q(\zeta_r) \sum_{\zeta_\ell \in \mathcal{B}} q^{-1}(\zeta_\ell) \zeta_\ell x. \quad \square
\end{aligned}$$

Corollary 4.7. *The basis described in Lemma 4.5 is independent of the choice of \mathcal{C} .*

Proof. Let \mathcal{C} and \mathcal{C}' be two choices of Z -orbit representatives, as in Definition 4.3. For a given Z -orbit, let $gx \in \mathcal{C}$ and $g'x \in \mathcal{C}'$ (with $g, g' \in \mathrm{GL}(2, \mathbb{Z})$) be the orbit representatives. Note that they will have the same representative $x \in \mathcal{A}$, since they are in the same $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit. Since gx and $g'x$ are in the same Z -orbit, for some $\zeta \in Z$ we have $\zeta g'x = gx$, so that $\zeta g^{-1}g'x = x$, and (choosing a $y \in Y$ representing x) we see that $\zeta g^{-1}g'y\alpha = y$ for some $\alpha \in K_{S,q}$. Hence, $\zeta r_y(\alpha) \in \mathrm{GL}(2, \mathbb{Z})$, and by Lemma 3.14 we see that $q(\zeta) = 1$. The corresponding basis elements then differ by a factor of ζ , so that by Lemma 4.6 they differ by a scalar factor of $q(\zeta) = 1$. \square

Lemma 4.8. *If we consider $M_{S,q}$ as a $\mathrm{GL}(2, \mathbb{Z})$ -module, it is a sum of induced modules. In fact, we have isomorphisms of $\mathrm{GL}(2, \mathbb{Z})$ -modules defined by*

$$e : \bigoplus_{x \in \mathcal{A}} \mathbb{F}[\mathrm{GL}(2, \mathbb{Z})] \otimes_{\mathbb{F}\Gamma_x} \mathbb{F} \rightarrow M_{S,q},$$

and

$$f : M_{S,q} \rightarrow \bigoplus_{x \in \mathcal{A}} \mathbb{F}[\mathrm{GL}(2, \mathbb{Z})] \otimes_{\mathbb{F}\Gamma_x} \mathbb{F},$$

such that e and f are inverses of each other.

Remark 4.9. These isomorphisms depend on the choice of \mathcal{A} , which we suppress from the notation.

Proof. On basis elements of the form $g \otimes 1 \in \mathbb{F}[\mathrm{GL}(2, \mathbb{Z})] \otimes_{\mathbb{F}\Gamma_x} \mathbb{F}$ for $x \in \mathcal{A}$, we define e as

$$e(g \otimes 1) = \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r g x,$$

and extend linearly. Since any $g \in \Gamma_x$ acts trivially on x , this is well-defined, and it is clearly a homomorphism of $\mathrm{GL}(2, \mathbb{Z})$ -modules.

We define f on basis elements corresponding to $gx \in \mathcal{C}$ (with $x \in \mathcal{A}$ and $g \in \mathrm{GL}(2, \mathbb{Z})$) by

$$f \left(\sum_{\zeta \in \mathcal{B}} q^{-1}(\zeta) \zeta gx \right) = g \otimes 1 \in \mathbb{F} \mathrm{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F} \Gamma_x} \mathbb{F} \subset \bigoplus_{x \in \mathcal{A}} \mathbb{F} \mathrm{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F} \Gamma_x} \mathbb{F},$$

and extending linearly. By Lemma 4.5, this gives a well defined \mathbb{F} -linear map from $M_{S,q}$ to

$$\bigoplus_{x \in \mathcal{A}} \mathbb{F} \mathrm{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F} \Gamma_x} \mathbb{F}.$$

One sees that on basis elements, e and f are inverses. Hence, they are inverses of each other as \mathbb{F} -vector space maps. Since e is a $\mathrm{GL}(2, \mathbb{Z})$ -module map, so is f and thus both are $\mathrm{GL}(2, \mathbb{Z})$ -module isomorphisms. \square

5. HOMOLOGY WITH COEFFICIENTS IN $M_{S,q}$ AND HECKE OPERATORS

In this section, we will fix an element $x_0 \in X_{S,q}$, and choose a set \mathcal{A} of $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit representatives in $X_{S,q}$ that contains x_0 .

As a consequence of Lemma 4.8, we have that

$$M_{S,q} \cong \bigoplus_{x \in \mathcal{A}} \mathrm{Ind}_{\Gamma_x}^{\mathrm{GL}(2, \mathbb{Z})} \mathbb{F}.$$

Hence, by Shapiro's lemma, we have

$$H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}) \cong \bigoplus_{x \in \mathcal{A}} H_1(\Gamma_x, \mathbb{F}).$$

Since Γ_x is infinite cyclic, we have

$$H_1(\Gamma_x, \mathbb{F}) \cong H^0(\Gamma_x, \mathbb{F}) \cong \mathbb{F}.$$

For each $x \in \mathcal{A}$, we choose a generator z_x for $H_1(\Gamma_x, \mathbb{F})$, as in Definition 2.14.

We now examine an individual Hecke operator. Let $s \in S_0$, and let E be a collection of single coset representatives for $\mathrm{GL}(2, \mathbb{Z}) s \mathrm{GL}(2, \mathbb{Z})$, so that

$$\mathrm{GL}(2, \mathbb{Z}) s \mathrm{GL}(2, \mathbb{Z}) = \coprod_{s_\alpha \in E} \mathrm{GL}(2, \mathbb{Z}) s_\alpha.$$

Then E is a finite set. At this point, the s_α may be altered by left-multiplication by elements of $\mathrm{GL}(2, \mathbb{Z})$. We now adjust the elements of E to make the computation of Hecke operators easier.

Because of our choice of \mathcal{A} , we have that $x_0 \in \mathcal{A}$. For convenience in what follows, we will write $\Gamma_{x_0} = \Gamma_0$. Recall that \mathcal{A} is a collection of $Z \mathrm{GL}(2, \mathbb{Z})$ -representatives of $X_{S,q}$. Hence, we may write any element of $X_{S,q}$ as $\zeta \gamma x$ for some $x \in \mathcal{A}$, $\zeta \in \mathcal{B}$ and $\gamma \in \mathrm{GL}(2, \mathbb{Z})$. Suppose that for $s_\alpha \in E$, we have

$$s_\alpha x_0 = \zeta \gamma x.$$

We will then adjust s_α , replacing it by $\gamma^{-1} s_\alpha$, and denote the corresponding ζ by ζ_α , and the corresponding x by x_α , so that we have

$$s_\alpha x_0 = \zeta_\alpha x_\alpha.$$

We now fix this choice of E .

For a given $\zeta \in Z$ and $x \in \mathcal{A}$, we define

$$E(\zeta x) = \{s_\alpha \in E : s_\alpha x_0 = \zeta x\}.$$

Since each $s_\alpha x_0$ is of the form ζx , we see easily that E is a disjoint union of the $E(\zeta x)$ as ζ runs through \mathcal{B} and x runs through \mathcal{A} (and $E(\zeta x)$ is empty for all but finitely many $x \in \mathcal{A}$).

Now, let $\zeta \in \mathcal{B}$, $x \in \mathcal{A}$, and (to avoid triviality) assume that for some $s_\alpha \in \mathcal{A}$, we have $s_\alpha x_0 = \zeta x$. We will then define $W_{\zeta x}$ to be the set of all elements $g \in \mathrm{GL}(2, \mathbb{Z}) s \mathrm{GL}(2, \mathbb{Z})$ such that $g x_0 = \zeta x$. This set is nonempty, and stable under right multiplication by Γ_0 and under left multiplication by Γ_x . Hence, we may write it as a disjoint union of double cosets

$$W_{\zeta x} = \coprod_{t \in T_{\zeta x}} \Gamma_x t \Gamma_0,$$

for some subset $T_{\zeta x} \subseteq W_{\zeta x}$.

For each $t \in T_{\zeta x}$, we may choose a set $B_{\zeta x, t}$ of single coset representatives and write the double coset $\Gamma_x t \Gamma_0$ as a disjoint union of single cosets

$$\Gamma_x t \Gamma_0 = \coprod_{t_\beta \in B_{\zeta x, t}} \Gamma_x t_\beta.$$

Lemma 5.1. *With the notation described above,*

- (1) *E is a disjoint union of the $E(\zeta x)$ for $\zeta \in \mathcal{B}$ and $x \in \mathcal{A}$.*
- (2) *For each $t \in T_{\zeta x}$, we may choose $B_{\zeta x, t}$ to be a subset of E .*
- (3) *With this choice, $E(\zeta x)$ is the disjoint union of the $B_{\zeta x, t}$ over all $t \in T_{\zeta x}$, and $B_{\zeta x, t} = \Gamma_x t \Gamma_0 \cap E(\zeta x)$.*

Proof. First, note that because ζ is central, $\Gamma_{\zeta x} = \Gamma_x$ for any $x \in X_{S, q}$.

We have seen (1) above.

For (2), choose $t \in T_{\zeta x}$, and let $\Gamma_x u$ be any single coset in $\Gamma_x t \Gamma_0$. Then $u x_0 = \zeta x$, and u is in some single coset $\mathrm{GL}(2, \mathbb{Z}) s_\alpha$ for some $s_\alpha \in E$. Then $u = g s_\alpha$ for some $g \in \mathrm{GL}(2, \mathbb{Z})$. We then have that

$$\zeta x = u x_0 = g s_\alpha x_0 = g \zeta_\alpha x_\alpha.$$

This implies that x and x_α are in the same $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit, and hence equal (since both come from \mathcal{A}). Hence, $g \in \Gamma_x$, and consequently $\zeta_\alpha x = \zeta x$. Therefore $\zeta_\alpha \zeta^{-1} \in Z_{S, q}$ and since $\zeta_\alpha, \zeta \in \mathcal{B}$, it follows that $\zeta_\alpha = \zeta$. Hence, we have $s_\alpha \in E(\zeta x)$, and $\Gamma_x u = \Gamma_x s_\alpha$, so we see that we may take the coset representative of $\Gamma_x u$ to be $s_\alpha \in E$.

For (3), we first note that any coset $\Gamma_x t$ for $t \in T_x$ contains exactly one s_α : part (2) shows that it contains at least one; if it contained two, say s_α and s_δ , then they would differ by left multiplication by $\gamma \in \Gamma_x \subseteq \mathrm{GL}(2, \mathbb{Z})$, which would imply $\mathrm{GL}(2, \mathbb{Z}) s_\alpha = \mathrm{GL}(2, \mathbb{Z}) s_\delta$ and therefore $s_\alpha = s_\delta$. Hence, it suffices to show that each $s_\alpha \in E(\zeta x)$ is contained in $B_{\zeta x, t}$ for some $t \in T_{\zeta x}$. This is, however, clear, since such an s_α is contained in

$$W_{\zeta x} = \bigcup_{t \in T_{\zeta x}} \Gamma_x t \Gamma_0.$$

The last assertion is now clear. □

From this point on, we will take $B_{\zeta x, t} = \Gamma_x t \Gamma_0 \cap E(\zeta x)$.

We continue to keep a fixed x_0 . As $\zeta \in \mathcal{B}$, $x \in \mathcal{A}$, and $t \in T_{\zeta x}$ vary, finitely many $B_{\zeta x, t}$ will be nonempty. We denote these sets by B_1, \dots, B_J for $J \in \mathbb{Z}$ positive, and for $j \in \{1, \dots, J\}$ we write x_j , ζ_j , and t_j for the corresponding values of ζ , x ,

and t , respectively. In addition, we will write Γ_j for Γ_{x_j} . With this notation, we now have

$$t_j x_0 = \zeta_j x_j \quad \text{and} \quad s_\alpha x_0 = \zeta_j x_j$$

for all $s_\alpha \in B_j$.

We note that for $s_\alpha \in B_j$, there exist $\eta_\alpha \in \Gamma_j$ and $\delta_\alpha \in \Gamma_0$ such that $s_\alpha = \eta_\alpha t_j \delta_\alpha$.

Lemma 5.2. *Fix $j \in \{1, \dots, J\}$, and for each $s_\alpha \in B_j$, write $s_\alpha = \eta_\alpha t_j \delta_\alpha$ with $\delta_\alpha \in \Gamma_0$ and $\eta_\alpha \in \Gamma_j$. Let $d_j = |B_j|$. Then*

$$\Gamma_0 = \coprod_{s_\alpha \in B_j} (t_j^{-1} \Gamma_j t_j \cap \Gamma_0) \delta_\alpha$$

so that we have

$$[\Gamma_0 : t_j^{-1} \Gamma_j t_j \cap \Gamma_0] = d_j.$$

Proof. First we show that the displayed cosets are all different. Suppose $s_\alpha, s_\beta \in B_j$ with $\delta_\alpha \delta_\beta^{-1} = t_j^{-1} \gamma_j t_j \in t_j^{-1} \Gamma_j t_j \cap \Gamma_0$ for some $\gamma_j \in \Gamma_j$. Then $\delta_\alpha = t_j^{-1} \gamma_j t_j \delta_\beta$. Hence

$$\begin{aligned} \Gamma_j s_\alpha &= \Gamma_j \eta_\alpha t_j \delta_\alpha \\ &= \Gamma_j \eta_\alpha t_j (t_j^{-1} \gamma_j t_j \delta_\beta) \\ &= \Gamma_j t_j \delta_\beta \\ &= \Gamma_j \eta_\beta t_j \delta_\beta \\ &= \Gamma_j s_\beta, \end{aligned}$$

where we have used that $\eta_\alpha, \eta_\beta, \gamma_j \in \Gamma_j$. Hence, $s_\alpha = s_\beta$, and we see that the cosets $(t_j^{-1} \Gamma_j t_j \cap \Gamma_0) \delta_\alpha$ are pairwise disjoint for $s_\alpha \in B_j$.

It remains only to show that the union of the cosets is all of Γ_0 . Let $g \in \Gamma_0$. Since $\Gamma_j t_j \Gamma_0$ is a disjoint union of cosets of the form $\Gamma_j s_\alpha$ for some $\alpha \in B_j$, we have that $t_j g = \gamma_j s_\alpha$ for some $\gamma_j \in \Gamma_j$ and $s_\alpha \in B_j$. Then, since $s_\alpha = \eta_\alpha t_j \delta_\alpha$, for $\eta_\alpha \in \Gamma_j$ and $\delta_\alpha \in \Gamma_0$, we have $t_j g = \gamma_j \eta_\alpha t_j \delta_\alpha$. Hence

$$g \delta_\alpha^{-1} = t_j^{-1} \gamma_j \eta_\alpha t_j \in t_j^{-1} \Gamma_j t_j \cap \Gamma_0,$$

so $g \in (t_j^{-1} \Gamma_j t_j \cap \Gamma_0) \delta_\alpha$. \square

The Hecke operator $T_s = \mathrm{GL}(2, \mathbb{Z}) s \mathrm{GL}(2, \mathbb{Z})$ acts in the usual way on the homology $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$ (see below), and hence, via the Shapiro isomorphism on the group

$$\bigoplus_{x \in \mathcal{A}} H_1(\Gamma_x, \mathbb{F}).$$

We now work out the details of the action on this latter group. To do this, we use the following lemma concerning transfers and corestrictions. This lemma is standard, and follows easily from [4, Sec. III.9].

Lemma 5.3. *Let A be an infinite cyclic group with generator a , and $B \subset A$ a subgroup of index d , and suppose that A acts trivially on \mathbb{F} . For a group G acting trivially on \mathbb{F} , we identify $H_1(G, \mathbb{F})$ canonically with $G^{ab} \otimes_{\mathbb{Z}} \mathbb{F}$.*

- (i) *The transfer map $\mathrm{tr} : H_1(A, \mathbb{F}) \rightarrow H_1(B, \mathbb{F})$ takes the generator $a \otimes 1$ to the generator $a^d \otimes 1$.*
- (ii) *The corestriction map $i : H_1(B, \mathbb{F}) \rightarrow H_1(A, \mathbb{F})$ (i.e. the map induced by the inclusion $B \subset A$) takes the generator $a^d \otimes 1$ to d times the generator $a \otimes 1$.*

With notation as above, we will use the techniques of [1] to write the Hecke operator T_s (given as a sum of actions of all the s_α) acting on a generator z_0 , as a sum of partial Hecke operators U_{t_j} given as a sum of actions of the $s_{\beta,j} \in B_j$, so that U_{t_j} maps $H_1(\Gamma_0, \mathbb{F})$ to $H_1(\Gamma_j, \mathbb{F})$.

More precisely, if F_\bullet is a resolution of \mathbb{F} by free $\mathbb{F}\mathrm{GL}(2, \mathbb{Q})$ -modules, then T_s on $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$ sends the class of a cycle (using an obvious notation for elements of F_\bullet and $M_{S,q}$) $\sum_f f \otimes_{\mathbb{F}\mathrm{GL}(2, \mathbb{Z})} m(f)$ to the class of $\sum_\alpha \sum_f s_\alpha f \otimes_{\mathbb{F}\mathrm{GL}(2, \mathbb{Z})} s_\alpha m(f)$. The partial Hecke operator U_{t_j} sends the class of a cycle $\sum_f f \otimes_{\Gamma_0} \lambda(f)$ to the class of $\sum_{s_{\beta,j} \in B_j} \sum_f s_{\beta,j} f \otimes_{\Gamma_j} \lambda(f)$.

The following lemma follows immediately from Theorem 3.1 in [1].

Lemma 5.4. *T_s composed with the Shapiro isomorphism equals $\sum_{j=1}^J U_{t_j}$.*

From now on we will also use T_s to stand for $\sum_{j=1}^J U_{t_j}$, depending on the context.

Next, we write the partial Hecke operator in terms of the transfer, corestriction, and an adjoint map.

Theorem 5.5. *Recall that $x_0 \in \mathcal{A}$, and let z_0 be the generator of $H^1(\Gamma_0, \mathbb{F})$ chosen in Definition 2.14. Then $U_{t_j} : H_1(\Gamma_0, \mathbb{F}) \rightarrow H_1(\Gamma_j, \mathbb{F})$ is given as the composition of the three maps*

$$H_1(\Gamma_0, \mathbb{F}) \rightarrow H_1(\Gamma_0 \cap s_j^{-1} \Gamma_j s_j, \mathbb{F}) \xrightarrow{\phi_j} H_1(s_j \Gamma_0 s_j^{-1} \cap \Gamma_j, \mathbb{F}) \rightarrow H_1(\Gamma_j, \mathbb{F}),$$

where the first map is the transfer, the second map is the map induced on homology by the pair of maps $(\mathrm{Ad} t_j, \tau_j)$, where $\mathrm{Ad} t_j$ is conjugation by t_j on the group, and τ_j is multiplication by $q(\zeta_j)$ on the coefficient module, and the third map is corestriction.

Proof. We take a resolution F_\bullet of \mathbb{F} by free $\mathbb{F}[\mathrm{GL}(2, \mathbb{Q})]$ -modules and let Z_0 be a cycle representing z_0 . The map in Shapiro's lemma taking $H_1(\Gamma_0, \mathbb{F})$ into $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$ sends Z_0 to $Z_0 \otimes e_*(I \otimes 1)$. Then, on the level of cycles, we have

$$T_s(Z_0 \otimes e_*(I \otimes 1)) = \sum_{j=1}^J \sum_{s_\beta \in B_j} s_\beta Z_0 \otimes s_\beta e_*(I \otimes 1).$$

If we let w_j be the composition of the three maps in the statement of the theorem, then using Lemma 5.4 we see that what we need to show is that

$$w_j(Z_0) \otimes e_*(I \otimes 1) = \sum_{s_\beta \in B_j} s_\beta Z_0 \otimes s_\beta e_*(I \otimes 1).$$

On the level of cycles, using Lemma 5.2, the transfer of Z_0 is

$$\sum_{s_\beta \in B_j} \delta_\beta Z_0,$$

where we recall that for $s_\beta \in B_j$, we have written $s_\beta = \eta_\beta t_j \delta_\beta$, with $\eta_\beta \in \Gamma_j$ and $\delta_\beta \in \Gamma_0$. Applying ϕ_j yields

$$q(\zeta_j) \sum_{s_\beta \in B_j} (t_j \delta_\beta t_j^{-1})(t_j Z_0).$$

Finally, applying the corestriction gives

$$q(\zeta_j) \sum_{s_\beta \in B_j} t_j \delta_\beta Z_0 = q(\zeta_j) \sum_{s_\beta \in B_j} s_\beta Z_0.$$

We see that it suffices to show that $q(\zeta_j)e_*(I \otimes 1) = s_\beta e_*(I \otimes 1)$.
We have that

$$\begin{aligned}
 s_\beta e_*(I \otimes 1) &= s_\beta \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r x_0 \\
 &= \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r s_\beta x_0 \\
 &= \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r \zeta_j x_j \\
 &= \zeta_j \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r x_j \\
 &= q(\zeta_j) \sum_{\zeta_r \in \mathcal{B}} q(\zeta_r^{-1}) \zeta_r x_j \\
 &= q(\zeta_j) e_*(I \otimes 1).
 \end{aligned}$$

□

We now apply Theorem 5.5 to compute $U_{t_j}(z_0)$.

Corollary 5.6. *The partial Hecke operator U_{t_j} in Theorem 5.5 satisfies*

$$U_{t_j}(z_0) = e_j q(\zeta_j) z_j$$

where $e_j = [\Gamma_j : t_j \Gamma_0 t_j^{-1} \cap \Gamma_j]$.

Proof. By definition, $d_j = [\Gamma_0 : t_j^{-1} \Gamma_j t_j \cap \Gamma_0]$ and $e_j = [\Gamma_j : t_j \Gamma_0 t_j^{-1} \cap \Gamma_j]$. Hence $g_0^{d_j}$ is a generator of $\Gamma_0 \cap t_j^{-1} \Gamma_j t_j$, and $g_j^{d_j}$ is a generator of $t_j \Gamma_0 t_j^{-1} \cap \Gamma_j$. Considering $H_1(\Gamma_0, \mathbb{F})$ as $\Gamma_0 \otimes_{\mathbb{Z}} F$ we have $z_0 = g_0 \otimes 1$. Hence, by Lemma 5.3(i), the transfer takes z_0 to $g_0^{d_j} \otimes 1$, which is then mapped to $g_j^{d_j} \otimes q(\zeta_j)$ by ϕ_j . Finally, by Lemma 5.3(ii), the corestriction maps this to $e_j(g_j \otimes q(\zeta_j)) = e_j q(\zeta_j)(g_j \otimes 1) = e_j q(\zeta_j) z_j$. □

We now compute the value of e_j . Recall that $t_j x_0 = \zeta_j x_j$. For $j = 0, \dots, J$, choose $y_j \in Y$ such that x_j is represented by y_j , and recall that Γ_j is the stabilizer of x_j in $\mathrm{GL}(2, \mathbb{Z})$ and ϵ is the fundamental unit of \mathfrak{D} which we chose at the beginning of Section 2. Let $g_j \in \Gamma_j$ and $m_j \in \mathbb{Z}$ be defined as in Definition 2.12 (with $H = K_{S,q}$). Then g_j is a generator of Γ_j and $m_j > 0$. Set $\delta_j = \pm \epsilon^{m_j}$, where the sign is chosen so that $g_j y_j = y_j \delta_j$ and $\delta_j \in K_{S,q}$.

Lemma 5.7. *With notation as above,*

$$e_j = \mathrm{LCM}(m_0, m_j)/m_j, \quad d_j = \mathrm{LCM}(m_0, m_j)/m_0,$$

and $e_j m_j = d_j m_0$.

Proof. First, $g_0 y_0 = y_0 \delta_0$. Since $t_j x_0 = \zeta_j x_j$, we have $t_j y_0 = \alpha_j \zeta_j y_j$ for some $\alpha_j \in K_{S,q}$. Hence, $t_j^{-1} y_j = \alpha_j^{-1} \zeta_j^{-1} y_0$.

It follows that $t_j g_0 t_j^{-1} y_j = y_j \delta_0$. In addition, $g_j y_j = y_j \delta_j$, g_0 generates Γ_0 , and g_j generates Γ_j .

We may choose a generator h of $\Gamma_j \cap t_j \Gamma_0 t_j^{-1}$, so that h will be the smallest power of $t_j g_0 t_j^{-1}$ that is contained in Γ_j . This power must be the smallest positive integer k such that δ_0^k is a power of δ_j . Since $\delta_0, \delta_j \in K_{S,q}$ and $-1 \notin K_{S,q}$, we

see that k will be the smallest positive integer such that km_0 is a multiple of m_j . Hence, $km_0 = \text{LCM}(m_0, m_j)$, and we see that

$$hy_j = y_j \left(\pm \epsilon^{\text{LCM}(m_0, m_j)} \right).$$

It follows that

$$e_j = \text{LCM}(m_0, m_j)/m_j.$$

Reversing the roles of Γ_0 and Γ_j and switching t_j and t_j^{-1} , we obtain

$$d_j = \text{LCM}(m_0, m_j)/m_0. \quad \square$$

6. ELEMENTS OF $H^1(\text{GL}(2, \mathbb{Z}), M_{S,q}^*)$ INTERPRETED AS FUNCTIONS ON LATTICES

We now interpret the cohomology of the dual of $M_{S,q}$ as a collection of functions on a space of lattices.

Definition 6.1. Let Φ be a function from lattices in K to \mathbb{F} . We will say that Φ is q -homogeneous if $\Phi(\alpha L) = q(\alpha I)\Phi(L)$ for all $\alpha \in \mathbb{Q}^\times \cap K_{S_0}$ and all lattices in L .

We will say that Φ is $K_{S,q}$ -invariant if $\Phi(\alpha L) = \Phi(L)$ for all $\alpha \in K_{S,q}$ and all lattices L .

Remark 6.2. Note that since q is trivial on $K_{S,q}$, a function Φ can be both q -homogeneous and $K_{S,q}$ -invariant. In addition, we note that since K is a real quadratic field, $q(-I) = 1$. If this were not the case, the fact that $-L = L$ for any lattice L in K would force all q -homogeneous functions to be identically 0.

Lemma 6.3. *There is an isomorphism between $H^1(\text{GL}(2, \mathbb{Z}), M_{S,q}^*)$ and the vector space of \mathbb{F} -valued functions on lattices in K that are q -homogeneous and $K_{S,q}$ -invariant.*

Proof. Choose a set \mathcal{A} of representatives of the $Z\text{GL}(2, \mathbb{Z})$ -orbits in $X_{S,q}$. This choice of \mathcal{A} yields an isomorphism of $\text{GL}(2, \mathbb{Z})$ -modules

$$f : M_{S,q} \rightarrow \bigoplus_{x \in \mathcal{A}} \mathbb{F} \text{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F}\Gamma_x} \mathbb{F}.$$

This induces an isomorphism (via Shapiro's Lemma)

$$\begin{aligned} H_1(\text{GL}(2, \mathbb{Z}), M_{S,q}) &\cong \bigoplus_{x \in \mathcal{A}} H_1(\text{GL}(2, \mathbb{Z}), \mathbb{F} \text{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F}\Gamma_x} \mathbb{F}) \\ &\cong \bigoplus_{x \in \mathcal{A}} H_1(\Gamma_x, \mathbb{F}) \\ &\cong \bigoplus_{x \in \mathcal{A}} \mathbb{F}. \end{aligned}$$

Using the natural duality between $H_1(\text{GL}(2, \mathbb{Z}), M_{S,q})$ and $H^1(\text{GL}(2, \mathbb{Z}), M_{S,q}^*)$, we see that determining an element of $H^1(\text{GL}(2, \mathbb{Z}), M_{S,q}^*)$ is the same as giving a function from \mathcal{A} to \mathbb{F} .

We now show that there is an isomorphism between the vector space of functions from \mathcal{A} to \mathbb{F} and the vector space of $K_{S,q}$ -invariant q -homogeneous functions on lattices in K .

Let h be any q -homogeneous $K_{S,q}$ -invariant function on lattices in K . Since every element in \mathcal{A} can be lifted uniquely to a $K_{S,q}$ -homothety class of lattices in K , h defines a function g on \mathcal{A} . Namely, given $x \in \mathcal{A}$, lift x to $y \in Y$ and set

$g(x) = h(\Lambda_y)$, where Λ_y is the lattice spanned by the entries of y . Since y is well-defined up to $K_{S,q}$ -homotheties and h is $K_{S,q}$ -invariant, this gives a well-defined function g .

Given a function g on \mathcal{A} and a lattice L in K , L corresponds (by choosing a basis $y = {}^t(a, b) \in Y$) to an element $x' \in X_{S,q}$, which lies in the $Z\mathrm{GL}(2, \mathbb{Z})$ -orbit of a unique $x \in \mathcal{A}$. Let $x' = \zeta\gamma x$, with $\zeta \in Z$ and $\gamma \in \mathrm{GL}(2, \mathbb{Z})$. Define $h(L) = q(\zeta)g(x)$. Note that if $x' = \zeta'\gamma'x$ with $\zeta' \in Z$ and $\gamma' \in \mathrm{GL}(2, \mathbb{Z})$, then we have $(\zeta^{-1}\zeta')(\gamma^{-1}\gamma')x = x$. Hence, $(\zeta^{-1}\zeta')(\gamma^{-1}\gamma') = r_y(\alpha)$ for some $\alpha \in K_{S,q}$. By Lemma 3.14, this implies that $q(\zeta) = q(\zeta')$, so h is well defined. Then h is a q -homogeneous $K_{S,q}$ -invariant function on lattices.

These two maps (taking h to g and g to h) are easily seen to be inverses, and preserve addition and scalar multiplication. \square

7. THE BRANCHED BRUHAT-TITS GRAPH AND THE LAPLACIAN

In order to construct functions on lattices that are eigenfunctions of the Hecke operators, we will use a modification of the Bruhat-Tits building [6, 14], in which we lift the Bruhat-Tits building to a finite branched cover.

For each prime ℓ unramified in K , let K_ℓ denote $K \otimes \mathbb{Q}_\ell$. Then K_ℓ is a two-dimensional vector space over \mathbb{Q}_ℓ .

Definition 7.1. If ℓ is inert, then K_ℓ is a quadratic field extension of \mathbb{Q}_ℓ . We fix the integral basis $\{1, \omega\}$ of K , and we identify K_ℓ with \mathbb{Q}_ℓ^2 by identifying 1 and ω with the standard basis elements $e_1, e_2 \in \mathbb{Q}_\ell^2$.

If ℓ splits in K , then $(\ell) = \lambda\lambda'$ for prime ideals λ, λ' in \mathfrak{O} lying over ℓ . Each of the completions K_λ and $K_{\lambda'}$ is then isomorphic to \mathbb{Q}_ℓ . Restricting these isomorphisms to K , we obtain two distinct Galois conjugate embeddings $i_\lambda, i_{\lambda'} : K \rightarrow \mathbb{Q}_\ell$. We then identify $K_\ell = K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ with \mathbb{Q}_ℓ^2 via the map taking

$$t \otimes 1 \mapsto (i_\lambda(t), i_{\lambda'}(t)).$$

We abbreviate the notation by writing $t \mapsto (t, t')$.

Definition 7.2. By a lattice in K_ℓ , we will mean a rank two \mathbb{Z}_ℓ -submodule of K_ℓ .

If L is a lattice in K , then $L_\ell = L \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is a lattice in K_ℓ .

Definition 7.3. Let ℓ be a prime, and n a positive integer. Denote the elements of \mathbb{Q}_ℓ^\times with ℓ -adic valuation divisible by n by V_n . We note that V_n is a subgroup of index n of \mathbb{Q}_ℓ^\times .

Definition 7.4. Let L_1 and L_2 be lattices in K_ℓ . We say that L_1 and L_2 are n -homothetic if $L_1 = \alpha L_2$ for some $\alpha \in V_n$. Then n -homothety is an equivalence relation, and we call an equivalence class an n -homothety class of lattices in K_ℓ .

Definition 7.5. Let n a positive integer, K a real quadratic field, and ℓ a prime unramified in K . The branched Bruhat-Tits graph \mathcal{T}_ℓ^n is the graph whose vertices are n -homothety classes of lattices in K_ℓ . Two vertices are joined by an edge if there are representative lattices L_1 and L_2 of the vertices such that $L_2 \subset L_1$ or $L_1 \subset L_2$ with index ℓ .

Remark 7.6. The Bruhat-Tits tree is a special case of the branched Bruhat-Tits graph in which $n = 1$. When $n = 1$, we may denote \mathcal{T}_ℓ^n by \mathcal{T}_ℓ . When $n > 1$, we will typically write vertices of \mathcal{T}_ℓ^n with a superscript n , i.e. $t^n \in \mathcal{T}_\ell^n$.

Definition 7.7. Let L be a lattice in K_ℓ . Denote the vertex of \mathcal{T}_ℓ^n represented by L by $\varpi(L)$. Denote the vertex of \mathcal{T}_ℓ represented by L by $\pi(L)$. Given a vertex $t^n \in \mathcal{T}_\ell^n$, there is a unique vertex $s \in \mathcal{T}_\ell$ containing t^n ; we write $s = \pi(t^n)$.

Note that for any lattice L with $\varpi(L) = t^n$, $\pi(t^n) = \pi(L)$. To keep our notation less cluttered, if L is a lattice in K_ℓ , we will often denote $\varpi(L)$ by L , as long as the context makes this usage clear.

Remark 7.8. We note that for any vertex $t \in \mathcal{T}_\ell$, there are exactly n vertices $t^n \in \mathcal{T}_\ell^n$ with $\pi(t^n) = t$. If L is a lattice in K_ℓ representing t , these n vertices of \mathcal{T}_ℓ^n are represented by

$$L, \ell L, \dots, \ell^{n-1} L.$$

Definition 7.9. If t is a vertex of \mathcal{T}_ℓ , we will call the set $\{t^n \in \mathcal{T}_\ell^n : \pi(t^n) = t\}$ the *fiber* of t and also the *fiber* of t^n for any t^n in that set.

Definition 7.10. A vertex t^n is *idealistic* if t^n is the n -homothety class of I_ℓ for some fractional ideal I of K .

We now review some facts about completions L_ℓ of lattices in K . Let ℓ be a prime of \mathbb{Q} .

By [15, V.2, Corollary to Theorem 2], the operations of sum and intersection of lattices in K commute with completion at ℓ . In addition, by [15, V.3, Theorem 2], a lattice L in K is determined by its set of completions L_w for all finite places w of \mathbb{Q} . In fact

$$L = \bigcap_w K \cap L_w.$$

Finally, completion at a finite place w of finitely generated \mathbb{Z} -modules is an exact functor [7, Theorem 7.2].

Applying these facts to fractional ideals of K , we note that if I is an ideal of \mathfrak{O} of norm prime to ℓ , then I_ℓ is an ideal of \mathfrak{O}_ℓ of index prime to ℓ , so $I_\ell = \mathfrak{O}_\ell$. In addition, multiplication of relatively prime ideals (i.e. intersection) commutes with completion at ℓ . Hence, for an ideal I , the completion I_ℓ depends only on the factors of I of ℓ -power norm.

Now suppose that $t^n \in \mathcal{T}_\ell^n$ is idealistic. Then we may assume that t^n is represented by an ideal I_ℓ , where I is an ideal with ℓ -power norm in \mathfrak{O} . If ℓ is inert in K , such an I must be principal, so I_ℓ is \mathbb{Q}_ℓ -homothetic to \mathfrak{O}_ℓ . Hence, t^n is idealistic if and only if $\pi(t^n)$ is represented by \mathfrak{O}_ℓ .

On the other hand, if ℓ splits in K , then $\ell\mathfrak{O} = \lambda\lambda'$, where λ, λ' are prime ideals of \mathfrak{O} lying over ℓ . We then see that $t^n \in \mathcal{T}_\ell^n$ is idealistic if and only if $\pi(t^n)$ is represented by an ideal of the form λ^k or $(\lambda')^k$. In particular, if $\pi(t_1^n) = \pi(t_2^n)$, then t_1^n and t_2^n are either both idealistic, or both nonidealistic.

Lemma 7.11. Let $L_1 \supset L_2$ be lattices in K_ℓ with $[L_1 : L_2] = \ell$. Let $t_1^n = \varpi(L_1) \in \mathcal{T}_\ell^n$. Then there are precisely two vertices $t_2^n, t_3^n \in \mathcal{T}_\ell^n$ with $\pi(t_2^n) = \pi(t_3^n) = \pi(L_2)$, such that there is an edge between the two pairs (t_1^n, t_2^n) and (t_1^n, t_3^n) . If we let t_2^n be represented by L_2 , then t_3^n is represented by $\ell^{-1}L_2$.

Proof. Clearly, if we take $t_2^n = \varpi(L_2)$ and $t_3^n = \varpi(\ell^{-1}L_2)$, we see that t_2^n and t_3^n are distinct and have the desired properties. It remains to show that there is no third vertex t_4^n , distinct from t_2^n and t_3^n , with $\pi(t_4^n) = \pi(L_2)$, and such that there is an edge between t_4^n and t_1^n .

Suppose that there is an edge between t_1^n and t_4^n . Then either there is a lattice L_3 representing t_4^n such that $L_1 \supset L_3$ and $[L_1 : L_3] = \ell$ or there is a lattice L_3 representing t_4^n such that $L_1 \subset L_3$ and $[L_3 : L_1] = \ell$.

Now suppose L_3 is homothetic to L_2 , say with $L_3 = \alpha L_2$, where $\alpha \in \mathbb{Q}_\ell^\times$.

If $L_1 \supset L_3$ has index ℓ , then we have that $\ell^{-1}L_2 \supset L_1$ has index ℓ and $L_1 \supset \alpha L_2$ has index ℓ . Hence, multiplying by ℓ , we see that $L_2 \subset \ell \alpha L_2$ with index ℓ^2 . This implies that $v_\ell(\alpha) = 0$, so that $\alpha L_2 = L_2$, so $t_4^n = t_2^n$.

On the other hand, if $L_1 \subset L_3$ with index ℓ , we have that $L_2 \subset \alpha L_2$ has index ℓ^2 . Hence $v_\ell(\alpha) = -1$, and we see that $\alpha L_2 = \ell^{-1}L_2$, so $t_4^n = t_3^n$. \square

Corollary 7.12. *Let $n \geq 1$, let t^n be a vertex in \mathcal{T}_ℓ^n , and let $t = \pi(t^n) \in \mathcal{T}_\ell$. Let $s \in \mathcal{T}_\ell$ be a neighbor of t . Then there are exactly two neighbors s_1^n and s_2^n of t^n in \mathcal{T}_ℓ^n with $\pi(s_1^n) = \pi(s_2^n) = s$. If L represents t^n , then exactly one of s_1^n and s_2^n is represented by a sublattice L' of L of index ℓ ; the other is represented by $\ell^{-1}L'$, which contains L with index ℓ .*

Definition 7.13. Let $n \geq 1$, let $t^n \in \mathcal{T}_\ell^n$ be a vertex represented by a lattice L in K_ℓ , and let $s_1^n, s_2^n \in \mathcal{T}_\ell^n$ be two neighbors of t^n with $\pi(s_1^n) = \pi(s_2^n)$. Call the neighbor represented by a sublattice of index ℓ in L a *downhill* neighbor of t^n ; call the other an *uphill* neighbor of t .

Definition 7.14. Let $t^n \in \mathcal{T}_\ell^n$. We define the *tier* of t^n to be the distance between $\pi(t^n)$ and $\pi(\mathfrak{O}_\ell)$ in \mathcal{T}_ℓ . A neighbor of t^n of higher tier than t^n will be called an *outer neighbor* of t^n ; a neighbor of lower tier will be called an *inner neighbor*.

Remark 7.15. Each $t^n \in \mathcal{T}_\ell^n$ has precisely $\ell + 1$ downhill neighbors and $\ell + 1$ uphill neighbors. The use of uphill and downhill matches our intuition; if s^n is a downhill neighbor of t^n , then t^n is an uphill neighbor of s^n .

Each vertex of positive tier has precisely ℓ downhill outer neighbors, and 1 downhill inner neighbor. It also has precisely ℓ uphill outer neighbors, and 1 uphill inner neighbor.

A vertex of tier 0 has only outer neighbors; $\ell + 1$ of them are uphill, and $\ell + 1$ are downhill.

There is a natural action of the group $\mathrm{GL}(2, \mathbb{Q}_\ell)$ on \mathbb{Q}_ℓ^2 , namely matrix multiplication with elements of \mathbb{Q}_ℓ^2 considered as column vectors. We transfer this action to K_ℓ via the identification that we have made between K_ℓ and \mathbb{Q}_ℓ^2 . The action of $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ is invertible, and preserves \mathbb{Q}_ℓ -linear combinations, so it maps bases of \mathbb{Q}_ℓ^2 to bases, maps lattices to lattices, and preserves n -homothety of lattices. Hence, multiplication by g defines a bijection from \mathcal{T}_ℓ^n to \mathcal{T}_ℓ^n .

Definition 7.16. Let $F(\mathcal{T}_\ell^n)$ be the set of \mathbb{F} -valued functions on the vertices of \mathcal{T}_ℓ^n .

Definition 7.17. The *Laplace operator* Δ_ℓ^n on $F(\mathcal{T}_\ell^n)$ is defined by

$$\Delta_\ell^n(f)(t^n) = \sum_{u^n} f(u^n),$$

where the sum runs over the $\ell + 1$ downhill neighbors $u^n \in \mathcal{T}_\ell^n$ of $t^n \in \mathcal{T}_\ell^n$.

In the next lemma, we describe how the coset representatives for a Hecke operator act on lattices. Recall Lemmas 5.1, 5.2 and 5.4 for the definition of the sets B_j , the coset representatives $s_{\beta,j}$, and the integers d_j . Note that these definitions depend

on a choice of an element $x_0 \in X_{S,q}$ and a choice of $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit representatives \mathcal{A} containing x_0 .

Lemma 7.18. *Let $x_0 \in X_{S,q}$ and let \mathcal{A} be a set of $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit representatives containing x_0 . Let x_0 be represented by $y = (a_0, b_0) \in Y$ with $a_0, b_0 \in \mathfrak{D}$, and let $L_0 = L(a_0, b_0)$ be the lattice generated by a_0 and b_0 . Let $s = \mathrm{diag}(\ell, 1)$ and let*

$$\mathrm{GL}(2, \mathbb{Z})s\mathrm{GL}(2, \mathbb{Z}) = \coprod_{\alpha} \mathrm{GL}(2, \mathbb{Z})s_{\alpha}$$

with the s_{α} chosen and partitioned as described in Section 5.

- (i) $\mathcal{L} = \{s_{\alpha}L_0\}$ consists of the $\ell + 1$ lattices of index ℓ contained in L_0 .
- (ii) \mathcal{L} is partitioned into the subsets

$$\mathcal{L}_j = \{s_{\alpha}L_0 \mid s_{\alpha} \in B_j\},$$

and $|\mathcal{L}_j| = d_j$.

- (iii) The same is true of the completions at ℓ : $\mathcal{L}_{\ell} = \{s_{\alpha}(L_0)_{\ell}\}$ consists of the $\ell + 1$ lattices of index ℓ contained in $(L_0)_{\ell}$, and these are partitioned into the subsets

$$\mathcal{L}_{\ell,j} = \{s_{\alpha}(L_0)_{\ell} \mid \alpha \in B_j\}.$$

and $|\mathcal{L}_{\ell,j}| = d_j$

Proof. (i) If

$$s_{\alpha} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_{\alpha} \\ b_{\alpha} \end{pmatrix},$$

then $s_{\alpha}L_0 = L(a_{\alpha}, b_{\alpha})$. Since s_{α} is an integral matrix of determinant ℓ , it is clear that $L(a_{\alpha}, b_{\alpha})$ has index ℓ in L_0 , and all sublattices of L_0 of index ℓ arise this way.

- (ii) Since $\{s_{\alpha}\}$ is partitioned by the sets B_j , it is clear that the lattices are partitioned as indicated.
- (iii) If L has index ℓ in L_0 , then the completion L_{ℓ} has index ℓ in $(L_0)_{\ell}$, since taking completions of finitely generated modules is an exact functor. Given two lattices $L \neq M$, each having index ℓ in L_0 , we note that for all places $w \neq \ell$, $L_w = M_w = (L_0)_w$. Since a lattice is determined by its completions at all finite places, we must have $L_{\ell} \neq M_{\ell}$.

□

Definition 7.19. Let $\phi_{\ell}^n \in F(\mathcal{T}_{\ell}^n)$, let $x_0 \in X_{S,q}$, and let \mathcal{A} be any set of $Z \mathrm{GL}(2, \mathbb{Z})$ -orbit representatives of $X_{S,q}$ containing x_0 . Define the sets B_j in terms of x_0 and \mathcal{A} as in Lemma 7.18. If, for all choices of \mathcal{A} and for all $y = {}^t(a_0, b_0) \in Y$ with $a_0, b_0 \in \mathfrak{D}$ representing x_0 , we have that ϕ_{ℓ}^n is constant on the set

$$\{(s_{\beta,j}L(a_0, b_0))_{\ell} \mid \beta = 1, \dots, d_j\}$$

of vertices of \mathcal{T}_{ℓ}^n , then we will say that ϕ_{ℓ}^n is *locally constant* relative to T_{ℓ} and x_0 .

If ϕ_{ℓ}^n is locally constant relative to T_{ℓ} and all $x_0 \in X_{S,q}$, then we say that ϕ_{ℓ}^n is *locally constant*.

We remark that the condition $a_0, b_0 \in \mathfrak{D}$ could be relaxed to $a_0, b_0 \in K$ without effect. This is true because for any pair $a_0, b_0 \in K$, there is an integer m such that $m^n a_0, m^n b_0 \in \mathfrak{D}$; then $L(a_0, b_0)$ and $L(m^n a_0, m^n b_0)$ are n -homothetic and hence define the same vertex of \mathcal{T}_{ℓ}^n .

Definition 7.20. Let $t_0^n \in \mathcal{T}_{\ell}^n$ be the vertex represented by the lattice \mathfrak{D}_{ℓ} .

Lemma 7.21. *The action of $\mathrm{GL}(2, \mathbb{Z}_\ell)$ on \mathcal{T}_ℓ^n permutes the vertices of \mathcal{T}_ℓ^n , fixes vertices of tier 0, and preserves edges (including whether the edge is uphill or downhill) and the tier of each vertex.*

Proof. Since the action of $\mathrm{GL}(2, \mathbb{Z}_\ell)$ is invertible, it is clear that the map it induces on vertices is a bijection. In addition, if $\gamma \in \mathrm{GL}(2, \mathbb{Z}_\ell)$, and $L_1 \subset L_2$ with index ℓ , then $\gamma L_1 \subset \gamma L_2$ with index ℓ , so edges are preserved (including whether the edge is uphill or downhill).

Since the action of $\mathrm{GL}(2, \mathbb{Z}_\ell)$ fixes \mathbb{Z}_ℓ^2 , which is identified with \mathfrak{O}_ℓ , it fixes vertices of tier 0. Since it preserves neighbors, a simple inductive argument shows that it maps each vertex to a vertex of the same tier. \square

Lemma 7.22. *Multiplication by the fundamental unit $\epsilon \in K \subset K_\ell$ induces a permutation on the vertices of \mathcal{T}_ℓ^n given (on the level of \mathbb{Z}_ℓ -lattices) by multiplication by a matrix in $\mathrm{GL}(2, \mathbb{Z}_\ell)$.*

Proof. Suppose that ℓ is inert in K . In this case (see Definition 7.1), we have identified $K_\ell = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \omega$ with \mathbb{Q}_ℓ^2 . Since multiplication by ϵ is \mathbb{Q} -linear on K it induces a \mathbb{Q}_ℓ -linear map on K_ℓ . Hence, multiplication by ϵ is represented by a matrix in $\mathrm{GL}(2, \mathbb{Q}_\ell)$. Since multiplication by ϵ is an automorphism of \mathfrak{O}_ℓ , and \mathfrak{O}_ℓ is identified with $\mathbb{Z}_\ell^2 \subset \mathbb{Q}_\ell^2$, this matrix has entries in \mathbb{Z}_ℓ , and since ϵ has norm ± 1 , the matrix must have determinant ± 1 , and we see that the matrix is in $\mathrm{GL}(2, \mathbb{Z}_\ell)$.

Now suppose that ℓ is split. Referring to Definition 7.1 again, we have identified K_ℓ with \mathbb{Q}_ℓ^2 , where $c \in K$ is identified with $(c, c') \in \mathbb{Q}_\ell^2$. Hence, multiplication by ϵ is represented by the matrix $\mathrm{diag}(\epsilon, \epsilon')$, which is in $\mathrm{GL}(2, \mathbb{Z}_\ell)$. \square

Lemma 7.23. *Let $\phi \in F(\mathcal{T}_\ell^n)$ be a function on the vertices of \mathcal{T}_ℓ^n . Assume that for every vertex $t^n \in \mathcal{T}_\ell^n$, ϕ is constant on the set of non-idealistic outer downhill neighbors u^n of t^n . Then ϕ is locally constant relative to T_ℓ and any $x_0 \in X_{S,q}$.*

Proof. Assume that ϕ satisfies the conditions of the lemma. Let $x_0 \in X_{S,q}$, choose any collection \mathcal{A} of orbit representatives containing x_0 , and choose any $y = {}^t(a_0, b_0)$ with $a_0, b_0 \in \mathfrak{O}$ representing x_0 . Partition the set $\{s_\alpha\}$ of coset representatives for the Hecke operator T_ℓ as in Lemma 7.18.

Let t^n be the vertex of \mathcal{T}_ℓ^n represented by $L_0 = L(a_0, b_0)$. For each set B_j , we wish to show that ϕ is constant on the set $\{(s_{\beta,j} L_0)_\ell | s_{\beta,j} \in B_j\}$. Choose any $s_{\beta,j}$ and let u^n be the downhill neighbor of t^n represented by $(L_1)_\ell$, where $L_1 = s_{\beta,j} L_0$. Then L_1 is homothetic to a lattice with a basis representing x_j . We now divide the proof into 3 cases.

Case 1. Suppose u^n is idealistic. Then L_1 is a fractional ideal of K , and is homothetic to a fractional ideal with basis representing $x_j \in \mathcal{A}$. Hence, $m_j = i^S$, so $d_j = 1$ by Lemmas 3.12 and 5.7. Hence, there is only one vertex on which ϕ must be constant.

Case 2. Suppose that u^n is the unique downhill inner neighbor of t^n . Recall from Theorem 2.8 that $\Gamma_0 = \Gamma x_0$ fixes x_0 and is generated by an element g_0 that acts on L_0 as multiplication by $\delta_0 = \pm \epsilon^{m_x}$ with the sign chosen so that $\delta_0 \in K_{S,q}$. From Theorem 2.8, we see that

$$g_0 L_0 = \delta_0 L_0 = L_0.$$

Since multiplication by δ_0 fixes L_0 , it also fixes $(L_0)_\ell = t^n$. Multiplication by δ_0 also fixes each element of the fiber of $t_0^n = \varpi(\mathfrak{O}_\ell)$, so it must fix the

unique downhill path from t^n to the fiber of t_0^n . Hence, multiplication by δ_0 must fix u^n .

Now, both L_1 and $\delta_0 L_1$ are sublattices of L_0 of index ℓ . Since both must represent u^n , we see that they are equal. Since $\delta_0 L_1 = L_1$, we see that $m_j | m_0$, so that $d_j = 1$. Hence, again, there is only one vertex on which ϕ must be constant.

Case 3. Suppose u^n is a nonidealistic outer downhill neighbor of t^n . By cases 1 and 2, no vertex in $\{(s_{\beta,j} L_0)_\ell | \beta \in B_j\}$ can be idealistic or a downhill inner neighbor of t_n . Hence, ϕ is constant (by hypothesis) on all the vertices in the desired set. \square

8. CONSTRUCTION OF FUNCTIONS ON LATTICES; COMPARISON BETWEEN THE LAPLACIAN AND A HECKE OPERATOR

Definition 8.1. Let $q : Z \rightarrow \mathbb{F}^\times$ be the character defined in Definition 3.7, and let ℓ be a prime of \mathbb{Z} that is unramified in K . We say that a function $f \in F(\mathcal{T}_\ell^n)$ is q -homogeneous (or just homogeneous, if q is understood) if, for all lattices L in K ,

$$f(\ell L) = q(\ell I) f(L).$$

Definition 8.2. For all finite places w of \mathbb{Q} unramified in K , let $n_w = 1$ if w is inert in K , let $n_w = 2$ if w splits in K . Fix a prime ℓ of \mathbb{Q} not dividing pdN , and let W be the set of all finite places of \mathbb{Q} not dividing ℓpdN . For $w \in W$, let $\phi_w \in F(\mathcal{T}_w^{n_w})$ denote a homogeneous function such that $\phi_w(\mathfrak{O}_w) = 1$. We view the functions ϕ_w as fixed by the context, and do not include them in the following notation for Φ . For any homogeneous $\phi_\ell \in F(\mathcal{T}_\ell^{n_\ell})$, define the function $\Phi(\phi_\ell)$ on lattices L in K by the formula

$$\Phi(\phi_\ell)(L) = \phi_\ell(L) \prod_{w \in W} \phi_w(L_w).$$

Lemma 8.3. *The infinite product in the definition makes sense and $\Phi(\phi_\ell)$ is q -homogeneous. The map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is \mathbb{F} -linear.*

Proof. For any given L , we have that $L_w = \mathfrak{O}_w$ for almost all w , so the product is actually finite. The linearity of the map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is clear. Now suppose $\alpha \in \mathbb{Q}^\times \cap K_{S_0}$ and L is a lattice. Then α is prime to pdN and factors as

$$\alpha = \ell^{f_\ell} \prod_{w \in W} w^{f_w}.$$

Then

$$\Phi(\phi_\ell)(\alpha L) = \phi_\ell(\alpha L) \prod_{w \in W} \phi_w(\alpha L_w) = \phi_\ell(\ell^{f_\ell} L) \prod_{w \in W} \phi_w(w^{f_w} L_w).$$

Since ϕ_ℓ and all the ϕ_w are homogeneous, this equals

$$q(\ell^{f_\ell} I) \phi_\ell(L) \left(\prod_{w \in W} q(w^{f_w} I) \right) \left(\prod_{w \in W} \phi_w(L_w) \right) = q(\alpha I) \Phi(\phi_\ell)(L). \quad \square$$

We now proceed to the main theorem of this section: the comparison between the Hecke operator and the Laplace operator.

By Lemma 2.5 and the fact that $K_{S,q}$ is unit-cofinite, we see that for any lattice $L \subseteq K$, there is a minimal positive integer m_L such that $\epsilon^{m_L} L = L$ and one of

$\pm \epsilon^{m_L} \in K_{S,q}$. If L_1 and L_2 are K^\times -homothetic lattices in K , it is clear that $m_{L_1} = m_{L_2}$. Set $m'_L = m_L/i^S$. By Theorem 2.8, if $L = L(a, b)$ and x is the image in $X_{S,q}$ of $y = {}^t(a, b)$ then $m_L = m_x$. Therefore, by Lemma 3.12, $i^S|m_L$ and m'_L is a positive integer.

Definition 8.4. Let $\psi_\ell \in F(\mathcal{T}_\ell^{n_\ell})$. We define the *transform* of ψ_ℓ to be the function $\hat{\psi}_\ell \in F(\mathcal{T}_\ell^n)$ given by the formula

$$\hat{\psi}_\ell(t^n) = m'_L \psi_\ell(t^n),$$

where L is any lattice in \mathfrak{O} of ℓ -power index, such that L_ℓ represents t^n .

Lemma 8.5. *Given $\psi_\ell \in F(\mathcal{T}_\ell^n)$, the transform $\hat{\psi}_\ell$ is well defined.*

Proof. We need to show that for $t^n \in \mathcal{T}_\ell^n$, the value of m_L does not depend on the lattice L chosen to represent t^n . Note that up to homothety by powers of ℓ^n , there is a unique lattice $\Lambda \subseteq K_\ell$ representing t^n . By [15, V.2, Theorem 2] there is a unique lattice $L \subseteq \mathfrak{O}$ of ℓ -power index such that $L_\ell = \Lambda$. Since Λ is uniquely defined up to homothety by powers of ℓ^n , so too is L . Finally, since homothety does not change the value of m_L , we see that m_L does not depend on the choice of L , so m'_L does not. \square

If $\psi_\ell(\mathfrak{O}_\ell) = 1$, then $\hat{\psi}_\ell(\mathfrak{O}_\ell) = 1$, since $m'_\mathfrak{O} = 1$. In addition, if \mathbb{F} has characteristic 0, then a function ψ_ℓ is determined by its transform; this fails if any m'_L is divisible by the characteristic of \mathbb{F} .

Lemma 8.6. *Let $\ell \nmid pdN$ be prime. If $\psi_\ell \in F(\mathcal{T}_\ell^n)$ is homogeneous, then $\hat{\psi}_\ell$ is also homogeneous.*

Proof. If $t^n \in \mathcal{T}_\ell^n$ is represented by L_ℓ , with L a lattice of ℓ -power index in \mathfrak{O} , then ℓt^n is represented by ℓL_ℓ . Since $m'_L = m'_{\ell L}$, we have

$$\hat{\psi}_\ell(\ell t^n) = m'_{\ell L} \psi_\ell(\ell t^n) = m'_L q(\ell I) \psi_\ell(t^n) = q(\ell I) \hat{\psi}_\ell(t^n). \quad \square$$

We now fix a set \mathcal{A}_0 of representatives of the $Z \mathrm{GL}(2, \mathbb{Z})$ -orbits in $X_{S,q}$. Recall from Lemma 6.3 that this choice fixes an isomorphism between the cohomology group

$$H^1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}^*)$$

and q -homogenous, $K_{S,q}$ -invariant functions on lattices.

Theorem 8.7. *Let $\ell \nmid pdN$ be prime. For each finite place $w \in W$, fix a homogeneous function $\phi_w \in F(\mathcal{T}_\ell^{n_w})$, as in Definition 8.2. Let $n = n_\ell$, and let $\psi_\ell \in F(\mathcal{T}_\ell^n)$ be homogeneous. Assume that $\Phi(\hat{\psi}_\ell)$ is $K_{S,q}$ -homothety invariant. (It will be q -homogeneous by Lemma 8.3.)*

As in Lemma 6.3 and its proof, view $\Phi(\hat{\psi}_\ell)$ as an element of

$$H^1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}^*) \cong H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})^*.$$

That is to say, view $\Phi(\hat{\psi}_\ell)$ as an \mathbb{F} -valued functional on $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$, via the pairing

$$\langle \Phi(\hat{\psi}_\ell), \bullet \rangle : H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}) \rightarrow \mathbb{F}.$$

If ψ_ℓ is locally constant relative to T_ℓ and x_0 , then

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = m_0 \langle \Phi(\Delta_\ell \psi_\ell), z_0 \rangle.$$

Proof. By Lemma 5.4, $T_\ell = \sum_{j=1}^J U_{t_j}$. By Corollary 5.6, for $1 \leq j \leq J$, we have

$$U_{t_j}(z_0) = e_j q(\zeta_j) z_j.$$

Then

$$\begin{aligned} \langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle &= \langle \Phi(\hat{\psi}_\ell), T_\ell z_0 \rangle \\ &= \sum_{j=1}^J q(\zeta_j) e_j \langle \Phi(\hat{\psi}_\ell), z_j \rangle \\ &= \sum_{j=1}^J q(\zeta_j) e_j m'_j \langle \Phi(\psi_\ell), z_j \rangle. \end{aligned}$$

We have $e_j m'_j = m'_0 d_j$ by Lemma 5.7, so

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = \sum_{j=1}^J q(\zeta_j) m'_0 d_j \langle \Phi(\psi_\ell), z_j \rangle.$$

Now, for a fixed j , we will analyze the term $\langle \Phi(\psi_\ell), z_j \rangle$. Recall the definition of the partial Hecke operators $U_{t,j}$ and of the matrices $s_{\beta,j}$ from the paragraphs before Lemma 5.4. Also recall the matrices t_j and the fact that $t_j x_0 = \zeta_j x_j$ from the paragraphs before Lemma 5.2.

Because Φ is invariant under $K_{S,q}$ -homothety, we may choose $a_j, b_j \in K$ so that ${}^t(a_j, b_j) \in Y$ represents $x_j \in X_{S,q}$. In fact, we choose $a_0, b_0 \in K$ so that ${}^t(a_0, b_0) \in Y$ represents x_0 , and then set

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \zeta_j^{-1} t_j \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

We then obtain

$$\begin{aligned} \langle \Phi(\psi_\ell), z_j \rangle &= \Phi(\psi_\ell)(L(a_j, b_j)) \\ &= \psi_\ell(L(a_j, b_j)_\ell) \prod_{w \in W} \phi_w(L(a_j, b_j)_w). \end{aligned}$$

Since t_j is an integral matrix with determinant ℓ , we know that $t_j \in \text{GL}(2, \mathfrak{O}_w)$ for all $w \in W$. Factor

$$\zeta_j^{-1} = \left(\ell^{f_{\ell,j}} \prod_{w \in W} w^{f_{w,j}} \right) I.$$

(By Definition 3.7, primes not in $W \cup \{\ell\}$ cannot divide the numerators or denominators of the diagonal entries of the matrix $\zeta_j \in Z$.)

Then $L(a_j, b_j)_w$ is the same as the lattice $w^{f_{w,j}} L(a_0, b_0)_w$. Set

$$c = \prod_{w \in W} \phi_w(L(a_0, b_0)_w).$$

Since each ϕ_w is homogeneous, we obtain

$$\langle \Phi(\psi_\ell), z_j \rangle = \psi_\ell(L(a_j, b_j)_\ell) c \prod_{w \in W} q(w^{f_{w,j}} I).$$

Hence, we see that

$$\begin{aligned}
\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle &= \sum_{j=1}^J q(\zeta_j) d_j m'_0 \langle \Phi(\psi_\ell), z_j \rangle \\
&= c m'_0 \sum_{j=1}^J q(\zeta_j) d_j \psi_\ell(L(a_j, b_j)_\ell) \prod_{w \in W} q(w^{f_{w,j}} I) \\
&= c m'_0 \sum_{j=1}^J q(\ell^{f_{\ell,j}} I) d_j \psi_\ell(L(a_j, b_j)_\ell),
\end{aligned}$$

where we have used the factorization of ζ_j^{-1} .

On the other hand, since ψ_ℓ is assumed to be locally constant with respect to T_ℓ and x_0 , and any $s_{\beta,j}$ takes any vertex to a downhill neighbor, we have that for each $s_{\beta,j}$,

$$\begin{aligned}
\psi_\ell(s_{\beta,j} L(a_0, b_0)_\ell) &= \psi_\ell(t_j L(a_0, b_0)_\ell) \\
&= \psi_\ell(\zeta_j L(a_j, b_j)_\ell) \\
&= \psi_\ell((\ell^{f_{\ell,j}})^{-1} L(a_j, b_j)_\ell) \\
&= q((\ell^{f_{\ell,j}})^{-1} I) \psi_\ell(L(a_j, b_j)_\ell),
\end{aligned}$$

since ψ_ℓ is homogeneous.

Hence, using the fact that $d_j = |B_j|$, we have that

$$\begin{aligned}
\langle \Phi(\Delta_\ell^n \psi_\ell), z_0 \rangle &= \Phi(\Delta_\ell^n \psi_\ell)(L(a_0, b_0)) \\
&= (\Delta_\ell^n(\psi_\ell))(L(a_0, b_0)_\ell) \prod_{w \in W} \phi_w(L(a_0, b_0)_w) \\
&= c(\Delta_\ell^n(\psi_\ell))(L(a_0, b_0)_\ell) \\
&= c \sum_{j=1}^J \sum_{s_{\beta,j} \in B_j} \psi_\ell(s_{\beta,j} L(a_0, b_0)_\ell) \\
&= c \sum_{j=1}^J d_j q((\ell^{f_{\ell,j}})^{-1} I) \psi_\ell(L(a_j, b_j)_\ell),
\end{aligned}$$

where we have used Lemma 7.18. Multiplying both sides of the last equality by m'_0 yields the assertion of the theorem, because q has order 2. \square

Corollary 8.8. *For any $x_0 \in \mathcal{A}_0$, let $z_0 \in H_1(\Gamma_{x_0}, \mathbb{F})$ be the corresponding homology generator, and let L_0 be a lattice corresponding to x_0 . Assume that ψ_ℓ is locally constant relative to T_ℓ and x_0 . Further, assume that $\Phi(\hat{\psi}_\ell)$ is q -homogeneous and $K_{S,q}$ -invariant, and that $(\Delta_\ell^n \psi_\ell)((L_0)_\ell) = \mu \psi_\ell((L_0)_\ell)$. Then*

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = \mu \langle \Phi(\hat{\psi}_\ell), z_0 \rangle.$$

Proof. From Theorem 8.7 and linearity we have

$$\begin{aligned}\langle \Phi(\hat{\psi}_\ell)T_\ell, z_0 \rangle &= m'_0 \langle \Phi(\Delta_\ell \psi_\ell), z_0 \rangle \\ &= m'_0 \langle \Phi(\mu \psi_\ell), z_0 \rangle \\ &= \mu \langle \Phi(m'_0 \psi_\ell), z_0 \rangle \\ &= \mu \langle \Phi(\hat{\psi}_\ell)T_\ell, z_0 \rangle.\end{aligned}\quad \square$$

Corollary 8.9. *Assume that ψ_ℓ is locally constant relative to T_ℓ and every $x \in X_{S,q}$, that $\Phi(\hat{\psi}_\ell)$ is q -homogeneous and $K_{S,q}$ -invariant, and that $\Delta_\ell \psi_\ell = \mu \psi_\ell$.*

Then $\Phi(\hat{\psi}_\ell)$ is an eigenclass for T_ℓ with eigenvector μ and it is an eigenclass for $T_{\ell,\ell}$ with eigenvector $\theta(\ell)$.

Proof. First, we show that $\Phi(\hat{\psi}_\ell) \neq 0$. By definition,

$$\Phi(\hat{\psi}_\ell)(L) = \hat{\psi}_\ell(L_\ell) \prod_{w \in W} \phi_w(L_w).$$

By construction, $\phi_w(\mathfrak{O}_w) = 1$ for every $w \in W$ and $\psi_\ell(\mathfrak{O}_\ell) = 1$. Since $m'_{\mathfrak{O}_\ell} = 1$, also $\hat{\psi}_\ell(\mathfrak{O}_\ell) = 1$. Therefore, $\Phi(\hat{\psi}_\ell)(\mathfrak{O}) = 1$.

For any $x \in \mathcal{A}_0$, write $z_x \in H^1(\Gamma_x, \mathbb{F})$ for the homology generator corresponding to x . By Corollary 8.8 and our hypothesis, for each $x \in \mathcal{A}_0$, we have

$$\langle \Phi(\hat{\psi}_\ell)T_\ell, z_x \rangle = \langle \mu \Phi(\hat{\psi}_\ell), z_x \rangle.$$

Since $\Phi(\hat{\psi}_\ell)$ is in the dual space to $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$, and $\{z_x : x \in \mathcal{A}_0\}$ spans $H_1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q})$, we are finished with T_ℓ .

As for $T_{\ell,\ell}$, its action is given by the double coset of the central element ℓI . So this is just a single coset, and its action on homology is given by the central character q on the coefficient module $M_{S,q}$. Since $q(\ell I) = \theta(\ell)$,

$$\langle \Phi(\hat{\psi}_\ell)T_{\ell,\ell}, z_x \rangle = \langle \Phi(\hat{\psi}_\ell), T_{\ell,\ell} z_x \rangle = \langle \Phi(\hat{\psi}_\ell), \theta(\ell) z_x \rangle = \langle \theta(\ell) \Phi(\hat{\psi}_\ell), z_x \rangle.$$

Hence, $\Phi(\hat{\psi}_\ell)T_{\ell,\ell} = \theta(\ell) \Phi(\hat{\psi}_\ell)$. \square

9. CONSTRUCTING LOCALLY CONSTANT EIGENFUNCTIONS

Recall that θ is the quadratic Dirichlet character associated to the real quadratic field K/\mathbb{Q} , and q is the character on Z defined by setting $q(rI) = \theta(r)$ for $r \in \mathbb{Z} \cap K_{S_0}$ and extending multiplicatively to Z . Since K is real quadratic, $q(-I) = \theta(-1) = 1$. Fix an \mathbb{F} -valued character χ on the group of ideals of K relatively prime to N for some positive integer N .

In this section, we will construct locally constant q -homogeneous functions ψ_ℓ^0 on $\mathcal{T}_\ell^{\nu_\ell}$ that are eigenfunctions of the Laplace operator with eigenvalues related to χ . We do this first for inert primes ℓ .

Theorem 9.1. *Let ℓ be a prime of \mathbb{Q} that is inert in K/\mathbb{Q} and does not equal the characteristic of \mathbb{F} . Then there is a locally constant q -homogeneous function $\psi_\ell^0 \in F(\mathcal{T}_\ell^2)$ that is an eigenvalue of the Laplace operator with eigenvalue 0 and satisfies $\psi_\ell^0(\mathfrak{O}_\ell) = 1$.*

Proof. We define ψ_ℓ^0 inductively.

For vertices of tier 0, we define $\psi_\ell^0(\mathfrak{O}_\ell) = 1$ and $\psi_\ell^0(\ell \mathfrak{O}_\ell) = \theta(\ell) = -1$. We see easily that ψ_ℓ^0 is homogeneous on the vertices of tier 0.

On vertices $t^2 \in \mathcal{T}_\ell^2$ of tier 1, we define $\psi_\ell^0(t^2) = 0$. Clearly ψ_ℓ^0 is θ -homogeneous on vertices of tier 1. In addition, since all downhill neighbors of a vertex of tier 0 have tier 1, we can now compute $\Delta_\ell^2(\psi_\ell^0)$ on vertices of tier 0; we find that its value is 0, as desired. Finally, ψ_ℓ^0 is constant on all downhill neighbors of vertices of tier 0.

On each vertex $t^2 \in \mathcal{T}_\ell^2$ of tier 2, let $u^2 \in \mathcal{T}_\ell^2$ be the unique uphill neighbor of t^2 of tier 1, and we let v^2 be the unique downhill neighbor of u^2 of tier 0. We define $\psi_\ell^0(t^2) = -\psi_\ell^0(v^2)/\ell$. Because the unique uphill neighbor of ℓt^2 of tier 1 is ℓu^2 , which has a unique downhill neighbor of tier 0 equal to ℓv^2 , we see that with this definition, ψ_ℓ^0 is homogeneous on vertices of tier 1. In addition, for any vertex u^2 of tier 1, ψ_ℓ^0 is constant on the downhill neighbors of u^2 of higher tier, since its value on such vertices depends only on its value on the unique downhill inner neighbor of u^2 . Finally, we have constructed ψ_ℓ^0 so that

$$\Delta_\ell^2(\psi_\ell^0)(u^2) = 0$$

for each vertex u^2 of tier 1.

We continue; for vertices $t^2 \in \mathcal{T}_\ell^2$ of odd tier, we define $\psi_\ell^0(t^2) = 0$. This guarantees that for vertices u^2 of even tier, $\Delta_\ell^2(\psi_\ell^0)(u^2) = 0$, and that ψ_ℓ^0 is constant on all downhill neighbors of u^2 of higher tier. Further, with this definition, $\psi_\ell^0(\ell t^2) = 0 = \theta(\ell)\psi_\ell^0(t^2)$ so that ψ_ℓ^0 is homogeneous on vertices of odd tier.

For a vertex $t^2 \in \mathcal{T}_\ell^2$ of positive even tier, let u^2 be the unique uphill inner neighbor of t^2 , and let v^2 be the unique downhill inner neighbor of u^2 . We define $\psi_\ell^0(t^2) = -\psi_\ell^0(v^2)/\ell$. Clearly ψ_ℓ^0 is constant on all downhill outer neighbors of u^2 (since its value on such neighbors depends only on its value on v^2). As in the case of tier 2, we see that $\psi_\ell^0(\ell t^2) = \theta(\ell)\psi_\ell^0(t^2)$, and $\Delta_\ell^2(\psi_\ell^0)(u^2) = \psi_\ell^0(v^2) + \ell(-\psi_\ell^0(v^2)/\ell) = 0$.

With this construction, we see that ψ_ℓ^0 is homogeneous, locally constant, and is an eigenfunction of Δ_ℓ^2 with eigenvalue 0. \square

Lemma 9.2. *For an inert prime ℓ , the function ψ_ℓ^0 defined above is $\mathrm{GL}(2, \mathbb{Z}_\ell)$ -invariant.*

Proof. The action of $\mathrm{GL}(2, \mathbb{Z}_\ell)$ fixes vertices of tier 0, and preserves uphill and downhill neighbors, and the tier of each vertex (Lemma 7.21). Since these relationships determine the values of ψ_ℓ^0 , the function is $\mathrm{GL}(2, \mathbb{Z}_\ell)$ -invariant. \square

For a prime ℓ that splits in K/\mathbb{Q} and does not divide N , we now prepare to construct a locally constant homogeneous function $\psi_\ell^0 \in F(\mathcal{T}_\ell^1)$ that is an eigenfunction of $\Delta_\ell = \Delta_\ell^1$. For the remainder of this section, we will assume that ℓ splits in K , that $(\ell) = \lambda\lambda'$ and that $\ell \nmid N$, so that $\chi(\lambda)$ and $\chi(\lambda')$ are defined. In this case, the function that we construct will depend not only on the real quadratic field K/\mathbb{Q} , but also on the character χ . Since we work in $\mathcal{T}_\ell^1 = \mathcal{T}_\ell$, the concepts of uphill and downhill neighbors coincide.

We begin by defining some terminology and notation for subsets of \mathcal{T}_ℓ .

Definition 9.3. We take \mathfrak{D}_ℓ as the basepoint of \mathcal{T}_ℓ and denote it by t_0 . A *descendant* of a vertex $t \in \mathcal{T}_\ell$ is a vertex $t_1 \neq t$ such that the path from t_0 to t_1 passes through t . Denote by $C(t)$ the set of all descendants t' of t such that every vertex of the path from t to t' except possibly t is non-idealistic, and let $\overline{C}(t) = C(t) \cup \{t\}$. We call $C(t)$ the *open cohort* of t , and $\overline{C}(t)$ the *closed cohort* of t .

Definition 9.4. A *simple chain* starting at a vertex $t \in \mathcal{T}_\ell$ is a collection C consisting of t and descendants of t such that for any pair $t', t'' \in C$, one of t', t'' is a descendant of the other. An *apartment* in \mathcal{T}_ℓ is a union of two infinite simple chains starting at a vertex t and having no other vertices in common.

Lemma 9.5. *Let t be an idealistic point in \mathcal{T}_ℓ .*

- (1) *If ℓ is inert, then $t = t_0$.*
- (2) *If $(\ell) = \lambda\lambda'$ splits and t is a distance $k > 0$ from t_0 , then $t = \lambda_\ell^k$ or $t = \lambda_\ell'^k$, and both of these points are a distance k from t_0 .*
- (3) *If (ℓ) splits and $k > 0$, then λ_ℓ^k and $\lambda_\ell'^k$ define distinct points in \mathcal{T}_ℓ .*
- (4) *No descendant of a non-idealistic point in \mathcal{T}_ℓ is idealistic.*
- (5) *The vertices of \mathcal{T}_ℓ are partitioned into the closed cohorts $\overline{C}(t_I)$ as $t_I = I_\ell$ runs over the idealistic points of \mathcal{T}_ℓ (where I is an ideal of \mathfrak{D} of ℓ -power norm.)*
- (6) *In the split case, the set of idealistic points of \mathcal{T}_ℓ form an apartment, namely*

$$\{\lambda_\ell^k | k > 0\} \cup \{t_0\} \cup \{\lambda_\ell'^k | k > 0\}.$$

Proof. In the discussion following Definition 7.10, we proved that the set of idealistic nodes of \mathcal{T}_ℓ is $\{t_0\}$ if ℓ is inert and $\{\lambda_\ell^k | k > 0\} \cup \{t_0\} \cup \{\lambda_\ell'^k | k > 0\}$ if ℓ is split. Since λ^k has index ℓ in λ^{k-1} , and similarly for the powers of λ' , (1) and (2) are now clear. As for (3), if $\lambda_\ell^k = \lambda_\ell'^k$, then $\lambda^k = \ell^m \lambda'^k$ for some integer m , which is absurd.

If ℓ is inert, (4) and (5) are obvious.

Assume then that ℓ splits. Then the idealistic point λ_ℓ^k is at the end of a path containing the nodes $t_0, \lambda_\ell, \dots, \lambda_\ell^k$. A similar statement holds for $\lambda_\ell'^k$. Since every non-idealistic node is a descendant of t_0 and \mathcal{T}_ℓ is a tree, no idealistic point can be a descendant of a non-idealistic point. Hence (4) holds.

For any node $u \in \mathcal{T}_\ell$ consider the path from t_0 to u (possibly of length 0.) Let t_I be the last idealistic point in this path. Then $u \in \overline{C}(t_I)$ is in the closed cohort of this idealistic point. If u were in the closed cohort of two distinct idealistic points, there would be a nontrivial loop in \mathcal{T}_ℓ . Hence, (5) holds.

Finally, (6) is clear, since the set of nonnegative powers of λ and of λ' each form a simple chain starting at t_0 . \square

Definition 9.6. Let N be a positive integer and c be an \mathbb{F} -valued multiplicative function on the group $I_K(N)$ of nonzero fractional ideals of K relatively prime to N . Fix a prime ℓ that does not divide N . Assume that c is trivial on the principal fractional ideal $\ell\mathfrak{D}$. Define $\hat{c} \in F(\mathcal{T}_\ell)$ by

$$\hat{c}(t) = \begin{cases} 0 & \text{if } t \text{ is non-idealistic,} \\ c(I) & \text{if } t = I_\ell, \text{ where } I \text{ is an ideal of } \ell\text{-power index in } \mathfrak{D}. \end{cases}$$

Lemma 9.7. *The function \hat{c} is well defined.*

Proof. Suppose I and J are both ideals of \mathfrak{D} of ℓ -power index, and that I_ℓ and J_ℓ are homothetic in K_ℓ by a power of ℓ .

If ℓ is inert, then I and J are both powers of ℓ . They are thus both principal, and we see that $c(I) = c(J) = c(\mathfrak{D})$.

If $(\ell) = \lambda\lambda'$ splits, then $I = \ell^a \mu$ and $J = \ell^b \nu$ for nonnegative integers a, b, q, r , and $\mu, \nu \in \{\lambda, \lambda'\}$. The fact that I_ℓ and J_ℓ are homothetic implies that $\mu = \nu$ and $a = b$, so I and J differ by a factor of ℓ^{q-r} . Since c is trivial on $\ell\mathfrak{D}$, $c(I) = c(J)$. \square

Let $t \in \mathcal{T}_\ell$. For any x in the open cohort $C(t)$ of t , all of the neighbors of x are in the closed cohort $\overline{C}(t)$. Hence, the Laplace operator Δ_ℓ defines a linear map from functions on $\overline{C}(t)$ to functions on $C(t)$.

Lemma 9.8. *Assume that ℓ is not equal to the characteristic of \mathbb{F} . Let $\mu \in \mathbb{F}$, and let t be an idealistic point of \mathcal{T}_ℓ with closed cohort $\overline{C}(t)$. Then there is a unique \mathbb{F} -valued function $\theta_{t,\mu}$ on $\overline{C}(t)$ with the following properties:*

- (i) $\theta_{t,\mu}(t) = 1$,
- (ii) $\theta_{t,\mu}(s) = 0$ for every $s \in C(t)$ that is distance 1 from t ,
- (iii) $\theta_{t,\mu}(s)$ depends only on ℓ , μ , and the distance from s to t ,
- (iv) $\Delta_\ell(\theta_{t,\mu})(s) = \mu\theta_{t,\mu}(s)$ for every $s \in C(t)$.

Proof. Define a sequence $a_k \in \mathbb{F}$ for $k \geq 0$ by the recurrence relation $a_0 = 1$, $a_1 = 0$, and for $k \geq 2$,

$$a_k = \frac{\mu a_{k-1} - a_{k-2}}{\ell}.$$

This clearly defines a unique sequence. For s a distance k from t in $\overline{C}(t)$, set $\theta_{t,\mu}(s) = a_k$. With this definition, $\theta_{t,\mu}$ satisfies conditions (i), (ii), and (iii).

Given a point $s \in C(t)$ a distance k from t , s has one neighbor a distance $k-1$ from t , and ℓ neighbors a distance $k+1$ from t . Hence

$$\begin{aligned} \Delta_\ell(\theta_{t,\mu})(s) &= a_{k-1} + \ell a_{k+1} \\ &= a_{k-1} + \ell \left(\frac{\mu a_k - a_{k-1}}{\ell} \right) \\ &= \mu a_k \\ &= \mu \theta_{t,\mu}(s), \end{aligned}$$

so $\theta_{t,\mu}$ satisfies condition (iv).

Conversely, if $\theta_{t,\mu}$ is a function on $\overline{C}(t)$ satisfying condition (iii), then for any s a distance k from t , we may define $a_k = \theta_{t,\mu}(s)$. If in addition $\theta_{t,\mu}$ satisfies conditions (i), (ii), (iv), the a_k satisfy the recurrence relation given above. The uniqueness of $\theta_{t,\mu}$ follows from the uniqueness of the sequence $\{a_k\}$. \square

Definition 9.9. Let $\mu \in \mathbb{F}$, and assume ℓ does not divide N and does not equal the characteristic of \mathbb{F} . We define $\psi_\ell^0 \in F(\mathcal{T}_\ell)$ by

$$\psi_\ell^0(s) = \hat{\chi}(t)\theta_{t,\mu}(s),$$

where $t \in \mathcal{T}_\ell$ is the unique idealistic vertex with $s \in \overline{C}(t)$.

Lemma 9.10. *Let $\mu \in F$ and assume that ℓ does not divide N and does not equal the characteristic of \mathbb{F} .*

- (1) $\psi_\ell^0(\mathfrak{O}_\ell) = 1$.
- (2) ψ_ℓ^0 is locally constant with respect to T_ℓ and any $x \in X_{S,q}$.
- (3) Let $\mu = \chi(\lambda) + \chi(\lambda')$. Then

$$\Delta_\ell \psi_\ell^0 = \mu \psi_\ell^0.$$

Proof. The first assertion is immediate from the definitions.

Let s be any vertex in \mathcal{T}_ℓ . We wish to show that ψ_ℓ^0 is constant on all non-idealistic outer downhill neighbors u of s . Then, by Lemma 7.23, part (2) will hold.

Let $s \in \bar{C}(t)$. Then any such u will be in $C(t)$. Since $\hat{\chi}(t)$ is constant for all points in $C(t)$, we need only show that $\theta_{t,\mu}(u)$ is constant for all such u . Letting the distance from t to s be $k-1$, the distance from t to u will be k . Hence, the desired constancy follows from Lemma 9.8(iii).

Now suppose that $s = t = I_\ell$ is idealistic. Then s has exactly two idealistic neighbors, namely $(\lambda I)_\ell$ and $(\lambda' I)_\ell$. The nonidealistic neighbors u of s are all in $C(t)$ and have distance 1 from t ; hence $\theta_{t,\mu}$ vanishes on them all. Hence

$$(\Delta_\ell \psi_\ell^0)(s) = \chi(\lambda I) + \chi(\lambda' I) = (\chi(\lambda) + \chi(\lambda'))\chi(I) = \mu \psi_\ell^0(t).$$

Finally, suppose that s is non-idealistic and belongs to the open cohort $C(t)$. Then

$$\begin{aligned} (\Delta_\ell \psi_\ell^0)(s) &= \sum_u \psi_\ell^0(u) \\ &= \sum_u \hat{\chi}(t) \theta_{t,\mu}(u) \\ &= \hat{\chi}(t) \sum_u \theta_{t,\mu}(u) \\ &= \hat{\chi}(t) (\Delta_\ell \theta_{t,\mu})(s) \\ &= \mu \hat{\chi}(t) \theta_{t,\mu}(s) \\ &= \mu \psi_\ell^0(s), \end{aligned}$$

by Lemma 9.8(iv), where the sums run over all neighbors u of s . \square

10. $K_{S,q}$ -INVARIANCE

Lemma 10.1. *Fix a prime ℓ that is unramified in K , and let $n = 1$ if ℓ splits in K and 2 if ℓ is inert. Let L be a \mathbb{Z} -lattice in K , and let $\alpha \in K^\times$. Let s^n be the vertex in \mathcal{T}_ℓ^n corresponding to L_ℓ , and let u^n be the vertex corresponding to $(\alpha L)_\ell$. Factor the fractional ideal $\alpha \mathfrak{D} = I_1 I_2$, where I_1 has norm a power of ℓ and I_2 is prime to ℓ .*

- (1) *There exists a matrix $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ depending only on α (independent of L), such that $u^n = gs^n$. If ℓ is inert, then $g = \ell^k g'$ with $k \in \mathbb{Z}$ and $g' \in \mathrm{GL}(2, \mathbb{Z}_\ell)$.*
- (2) *The vertex s^n is idealistic if and only if u^n is idealistic. If s^n corresponds to L_ℓ with L an ideal, then u^n corresponds to $(I_1 L)_\ell$.*
- (3) *Suppose ℓ is split. Assume that s^n is not idealistic, but lies in the open cohort $C(t)$ of the idealistic point $t^n = M_\ell$, where M is an ideal of ℓ -power norm. Then u^n lies in the open cohort $C(t_1^n)$, where $t_1^n = (I_1 M)_\ell$ and the distance between s^n and t^n is the same as the distance between u^n and t_1^n .*

Proof. (1) First, suppose that ℓ is inert. Via our identification of K_ℓ with \mathbb{Q}_ℓ^2 , multiplication by α is a \mathbb{Q}_ℓ -linear isomorphism from \mathbb{Q}_ℓ^2 to \mathbb{Q}_ℓ^2 ; hence, it is given by a matrix $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$. We can write $\alpha \in K_\ell$ as $\alpha = \ell^k \eta$ for some $k \in \mathbb{Z}$, and some unit $\eta \in \mathfrak{D}_\ell^\times$; multiplication by η is given by a matrix in $\mathrm{GL}(2, \mathbb{Z}_\ell)$.

Now assume that ℓ is split. In this case, we identify K_ℓ with \mathbb{Q}_ℓ^2 by mapping α to (α, α') . Then multiplication by α is defined by the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix},$$

which is in $\mathrm{GL}(2, \mathbb{Q}_\ell)$.

(2) L is a fractional ideal if and only if αL is a fractional ideal. If $L = MP$ with M a fractional ideal of ℓ -power norm, and P a fractional ideal prime to ℓ , then

$$(\alpha L)_\ell = (I_1 M)_\ell = (I_1 L)_\ell.$$

(3) Let $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ be the matrix from part (1) corresponding to multiplication by α . Multiplication by g is then an isometry of \mathcal{T}_ℓ that takes idealistic vertices to idealistic vertices, and non-idealistic vertices to non-idealistic vertices. Let R be a simple path from t^n to s^n whose only idealistic vertex is t^n . Then gR is a simple path from gt^n to u^n of the same length as R , whose only idealistic vertex is gt^n . Moreover, u^n lies in the open cohort $C(gt^n)$ where $gt^n = (I_1 M)_\ell$. \square

Theorem 10.2. *Let \mathbb{F} be a field of characteristic 0 or of finite characteristic not equal to two. If \mathbb{F} has characteristic 0, set $p = 1$, and otherwise let p be the characteristic of \mathbb{F} . Assume that χ is trivial on principal ideals generated by elements of $\mathbb{Q}^\times \cap K_{S_0}$. Also assume that χ is trivial on principal ideals generated by elements of $K_{S,q}$. Let Φ be the function from lattices in K to \mathbb{F} defined by*

$$\Phi(L) = \prod_{w \nmid pdN} \hat{\psi}_w^0(L_w).$$

Then $\Phi(\alpha L) = \Phi(L)$ for all $\alpha \in K_{S,q}$ and all lattices L in K .

Moreover, $\Phi(\alpha L) = q(\alpha I)\Phi(L)$ for all $\alpha \in \mathbb{Q}^\times \cap K_{S_0}$.

Proof. Let $L = L(c, d)$ be a lattice in K and let $\alpha \in K_{S,q}$. Note that $m'_L = m'_{\alpha L}$, since K^\times is commutative. Hence, there is a single integer m' , such that for each prime $w \nmid pdN$, we have

$$\hat{\psi}_w^0(L_w) = m' \psi_w^0(L_w)$$

and

$$\hat{\psi}_w^0((\alpha L)_w) = m' \psi_w^0((\alpha L)_w).$$

Assume first that w is inert in K . Then we may factor $\alpha \mathfrak{D}$ as

$$\alpha \mathfrak{D} = w^j I_2,$$

with I_2 a fractional ideal that is relatively prime to w . By Lemma 10.1(1), we have

$$(\alpha L)_w = w^j g L_w$$

for some $g \in \mathrm{GL}(2, \mathbb{Z}_w)$. Since ψ_w^0 is homogeneous and is $\mathrm{GL}(2, \mathbb{Z}_w)$ -invariant on \mathcal{T}_ℓ^2 (by Lemma 9.2), we have

$$\hat{\psi}_w^0((\alpha L)_w) = m' \psi_w^0(w^j g L_w) = m' q^*(w^j) \psi_w^0(L_w) = q^*(w^j) \hat{\psi}_w^0(L_w).$$

Now assume that w splits in K . Let $s \in \mathcal{T}_\ell$ be the vertex corresponding to L_w , and let u correspond to $(\alpha L)_w$.

If s is idealistic, so is u , and we see that

$$\psi_w^0(s) = \hat{\chi}(s) \theta_{s,\mu}(s) = \hat{\chi}(s) = \chi(L) = \chi(\alpha L) = \hat{\chi}(u) = \hat{\chi}(u) \theta_{u,\mu}(u) \psi_w^0(u).$$

If s is nonidealistic, then so is u , and $u = gs$ for some $g \in \mathrm{GL}(2, \mathbb{Q}_w)$. Suppose s lies in the open cohort $C(t)$ of the idealistic vertex t corresponding to I_w , where I is an ideal of w -power index in \mathfrak{D} . By Lemma 10.1(3), u is in the open cohort $C(t_1)$ of the idealistic point t_1 corresponding to $(I_1 I)_w$, where $\alpha \mathfrak{D} = I_1 I_2$, with I_1

having norm a power of w , and I_2 having norm relatively prime to w . In addition, the distance from s to t is the same as the distance from u to t_1 . Hence,

$$\hat{\chi}(t_1) = \chi(I_1 I) = \chi(I_1) \chi(I) = \chi(I_1) \hat{\chi}(t)$$

and

$$\theta_{t,\mu}(s) = \theta_{t_1,\mu}(u).$$

Therefore,

$$\begin{aligned} \hat{\psi}_w^0((\alpha L)_w) &= m' \psi_w^0(u) \\ &= m' \hat{\chi}(t_1) \theta_{t_1,\mu}(u) \\ &= m' \chi(I_1 I) \theta_{t_1,\mu}(u) \\ &= m' \chi(I_1) \hat{\chi}(t) \theta_{t,\mu}(s) \\ &= \chi(I_1) \hat{\psi}_w^0(L_w). \end{aligned}$$

In all of this, the fractional ideal I_1 depends on w ; we will call it $I_\alpha(w)$. Then $I_\alpha(w)$ is a product of powers of primes lying over w ; if w is inert, it is clear that $I_\alpha(w)$ is principal with a generator $\beta_\alpha(w)$ in $\mathbb{Q}^\times \cap K_{S_0}$, so that $\chi(I_\alpha(w)) = 1$.

Since $\alpha \in K_{S,q}$, α is relatively prime to pdN , so that

$$\alpha \mathfrak{D} = \prod_{w \nmid pdN} I_\alpha(w) = \left(\prod_{w \text{ inert}} I_\alpha(w) \right) \left(\prod_{w \text{ split}} I_\alpha(w) \right).$$

Setting $\beta = \prod_{w \text{ inert}} \beta_\alpha(w)$, we have

$$\beta \mathfrak{D} = \left(\prod_{w \text{ inert}} I_\alpha(w) \right).$$

Since $\alpha \in K_{S,q}$, $q^*(\alpha) = 1$. Because q^* depends only on inert prime factors, and the powers of inert primes dividing α and β are equal, we see that

$$1 = q^*(\alpha) = q^*(\beta).$$

In addition, we have that $\chi(\beta \mathfrak{D}) = 1$, since β is a product of powers of elements of $\mathbb{Q}^\times \cap K_{S_0}$, and we have assumed that χ is trivial on ideals generated by elements of $\mathbb{Q}^\times \cap K_{S_0}$. Hence, we see that

$$\prod_{w \text{ split}} I_\alpha(w)$$

is principal, with generator α/β , so

$$\prod_{w \text{ split}} \chi(I_\alpha(w)) = \frac{\chi(\alpha \mathfrak{D})}{\chi(\beta \mathfrak{D})} = \chi(\alpha \mathfrak{D}) = 1,$$

since we have assumed that χ is trivial on principal ideals generated by elements of $K_{S,q}$.

Hence, we obtain

$$\begin{aligned}
\Phi(\alpha L) &= \prod_{w \nmid pdN} \hat{\psi}_w^0((\alpha L)_w) \\
&= \left(\prod_{w \text{ inert}} \hat{\psi}_w^0((\alpha L)_w) \right) \left(\prod_{w \text{ split}} \hat{\psi}_w^0((\alpha L)_w) \right) \\
&= \left(\prod_{w \text{ inert}} q^*(\beta_\alpha(w)) \hat{\psi}_w^0((L)_w) \right) \left(\prod_{w \text{ split}} \chi(I_\alpha(w)) \hat{\psi}_w^0((L)_w) \right) \\
&= q^*(\beta) \left(\prod_{w \text{ split}} \chi(I_\alpha(w)) \right) \prod_{w \nmid pdN} \hat{\psi}_w^0((L)_w) \\
&= \Phi(L). \quad \square
\end{aligned}$$

Finally, if $\alpha \in \mathbb{Q}^\times \cap K_{S_0}$, it is a product of powers of primes not dividing pdN . We may thus assume that α is such a prime. The q -homogeneity of Φ then follows by Lemma 8.3 from the homogeneity of the individual $\hat{\psi}_w^0$ functions (see Theorem 9.1 for inert primes, and note that homogeneity is trivial for split primes).

11. GALOIS REPRESENTATIONS

We now define the Galois representations to which our main theorem below applies.

As before, we let K be a real quadratic field of discriminant d , cut out by the Dirichlet character θ . Let \mathbb{F} be a field of characteristic 0 (in which case we set $p = 1$) or a field of odd characteristic p , let G_K be the absolute Galois group of K (i.e. $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$), and let $\chi : G_K \rightarrow \mathbb{F}^\times$ be a character of G_K with finite image. By class field theory, we can think of χ as a character on the group of the nonzero fractional ideals of K relatively prime to N for some positive $N \in \mathbb{Z}$. Let L be the fixed field of the kernel of χ . Then L/K is Galois. We fix an $M \geq 3$ that divides pdN and define S_0 and S as in Definition 3.3.

We place the following conditions on the character χ .

- (1) χ is trivial on the principal fractional ideals of K generated by elements of $K_{S,q}$.
- (2) χ is trivial on the principal fractional ideals of K generated by elements of $\mathbb{Q}^\times \cap K_{S_0}$.
- (3) $[L : K]$ is odd.
- (4) L/\mathbb{Q} is Galois.

An example of such a χ would be any unramified character of G_K of odd order; such a character would be trivial on all principal fractional ideals of K , and L would be a subfield of the Hilbert class field of K and hence be Galois over \mathbb{Q} .

Let $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{F})$ be the induced representation

$$\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi.$$

Note that this representation will factor through $\mathrm{Gal}(L/\mathbb{Q})$. We have an exact sequence

$$1 \rightarrow \mathrm{Gal}(L/K) \rightarrow \mathrm{Gal}(L/\mathbb{Q}) \rightarrow \mathrm{Gal}(K/\mathbb{Q}) \rightarrow 1;$$

since $[L : K]$ is odd, this sequence splits, so there is an element τ of order 2 in $\text{Gal}(L/\mathbb{Q})$ mapping to the nonidentity element of $\text{Gal}(K/\mathbb{Q})$; we can lift it to an element $\tau \in G_{\mathbb{Q}}$, and we have that τ^2 is the identity modulo G_L .

With respect to a suitable basis, it is easy to see that for $g \in G_{\mathbb{Q}}$, we have the following:

(a) If $g \in G_K$, then

$$\rho(g) = \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi(g') \end{pmatrix},$$

where $g' = \tau^{-1}g\tau$.

(b) If $g \notin G_K$, then $g = h\tau$ for some $h \in G_K$, and

$$\rho(g) = \begin{pmatrix} 0 & \chi(h') \\ \chi(h\tau^2) & 0 \end{pmatrix},$$

where $h' = \tau^{-1}h\tau$.

If we now let g be a Frobenius element in $G_{\mathbb{Q}}$ for some prime ℓ of \mathbb{Q} not dividing pdN (so that ℓ is unramified in L/\mathbb{Q}), then we have the following two cases.

If ℓ splits in K and $\ell \nmid N$, then $g \in G_K$. If we write $\ell\mathfrak{D} = \lambda\lambda'$ with λ, λ' primes in K , then we may take g to be a Frobenius in G_K of λ ; a Frobenius of λ' will be g' . Hence, we have

$$\text{Tr}(\rho(g)) = \chi(g) + \chi(g') = \chi(\lambda) + \chi(\lambda'),$$

and

$$\det(\rho(g)) = \chi(g)\chi(g') = \chi(\lambda)\chi(\lambda') = \chi(\lambda\lambda') = \chi(\ell\mathfrak{D}) = 1$$

by condition (2) on the character χ .

On the other hand, if ℓ is inert in K and $\ell \nmid N$, write $g = h\tau$ as above. Then

$$\text{Tr}(\rho(g)) = 0$$

and $\det(\rho(g)) = -\chi(h\tau^2)\chi(h')$ with $h' = \tau^{-1}h\tau$. We note that g^2 is a Frobenius of $\ell\mathfrak{D}$ in G_K . Hence, we have

$$\det(\rho(g)) = -\chi(h\tau^2)\chi(h') = -\chi(h\tau^2h') = -\chi((h\tau)^2) = -\chi(g^2) = -\chi(\ell\mathfrak{D}) = -1,$$

where we have again used condition (2) on χ .

Note that in each case, when g is a Frobenius in $G_{\mathbb{Q}}$ of ℓ , we have $\det(\rho(g)) = \theta(\ell)$.

Now we check that ρ is even. Let $c \in G_{\mathbb{Q}}$ be a complex conjugation. Since c has order 2 and χ has odd order, $\chi(c) = \chi(\tau^{-1}c\tau) = 1$. From the formula in (a), since $c \in G_K$, $\rho(c)$ is the identity matrix.

Theorem 11.1. *Let K be a real quadratic field of discriminant d , let \mathbb{F} be a field of characteristic 0 or a finite field of odd characteristic. In the first case set $p = 1$ and in the second case let p be the characteristic of \mathbb{F} . Let $\chi : G_K \rightarrow \mathbb{F}^\times$ be a character with finite image. Let L be the fixed field of the kernel of χ and choose $N \in \mathbb{Z}$ so that L/K is unramified outside primes of K dividing N . Let $M \geq 3$, $S_0 = S_0(pdN)$, $S = S(M) \cap S_0$, θ the Dirichlet character cutting out K , q the character of Z determined by $q(rI) = \theta(r)$ for all $r \in \mathbb{Z} \cap K_{S_0}$, and $M_{S,q}$ the module defined in Definition 4.1. Assume*

- (1) χ is trivial on the principal fractional ideals of K generated by elements of $K_{S,q}$.
- (2) χ is trivial on the principal fractional ideals of K generated by elements of $\mathbb{Q}^\times \cap K_{S_0}$.

- (3) $[L : K]$ is odd.
- (4) L/\mathbb{Q} is Galois.

Then $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{F})$ given by $\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ is an even Galois representation, and is attached to a Hecke eigenclass in $H^1(\mathrm{GL}(2, \mathbb{Z}), M_{S,q}^*)$.

Proof. Given χ satisfying the conditions of the theorem, we define

$$\Phi(L) = \prod_{w \nmid pN} \hat{\psi}_w^0(L_w)$$

where ψ_w^0 is the function constructed in the proof of Theorem 9.1 for w inert in K and prime to pN , and the function defined by Definition 9.9 for w splitting in K and prime to pN .

By Theorem 10.2, Φ is $K_{S,q}$ -invariant and q -homogeneous. Hence, by Lemma 6.3 we may consider it as an element of $H^1(\mathrm{GL}(2, \mathbb{Q}), M_{S,q}^*)$. By Corollary 8.9, combined with Lemma 9.10 and Theorem 9.1 we see that for all ℓ unramified in L/\mathbb{Q} , Φ is an eigenvector for T_{ℓ} and $T_{\ell,\ell}$, and that the eigenvalues of T_{ℓ} match the trace of $\rho(\mathrm{Frob}_{\ell})$. The q -homogeneity of Φ shows that the eigenvalues of $T_{\ell,\ell}$ match the determinant of $\rho(\mathrm{Frob}_{\ell})$ for all ℓ unramified in L/\mathbb{Q} . Hence, Φ is attached to ρ . \square

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